# Watching gravitational waves

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**Abstract**

In this thesis the interaction of gravitational waves (GWs) with electromagnetic waves (EMWs) in a static magnetic background field is considered. This interaction is a consequence of the general relativistic Einstein field equations. These equations dictate the excitation of electromagnetic waves when a gravitational wave interferes with a static electromagnetic background field and a similar reverse process.

Such interactions can become effective, close to very energetic sources such as colliding neutron star binaries (i.e. gamma ray bursts), and quacking supernova remnants (magnetars), or in the large scale magnetic fields of the early universe. The former two sources have strong, localized and the latter weak, but extended magnetic fields. This is important because the energy transfer efficiency of GW ⇔ EMW conversions appears to be quadratically proportional to the background field strength and the extension of the interaction region.

All calculations in this thesis are done in a general relativistic, space+time, non-coordinate formalism. The reason for this is that in such *tedrad systems*, equations remain transparent and facilitate easy physical interpretation and connection to measurements.

In this framework, the conversion efficiency of GWs to light and vice versa in a vacuum is considered, first in an estimate and then in a more elaborate and general, exact calculation. The GW⇒EMW conversion is proposed as a possible indirect detection device for gamma ray bursts and magnetars. Also, the possibility is considered of explaining the small fluctuations in the cosmic background radiation by the conversion of GWs in the early universe to EMWs superposed on the homogeneous background radiation.

Next, to obtain more realistic results, the same process is examined in a thin plasma which leads to the same EMWs as generated in a vacuum. The importance of the presence of this plasma, though, is that it might damp the generated radio waves before they can travel over astromical distances, unless non-linear effects lead to higher frequencies of the EMWs. If radio waves with large enough frequencies are generated, gamma ray bursts and supernovæ might be detectable with (space based) radio detectors such as the proposed *Astronomical Low Frequency Array* (ALFA) as well as with GW detectors such as *Laser Interferometer Gravitational wave Observatory* (LIGO) with a event rate of as many as a few per year in our local galaxy group and the Virgo cluster.

In the last chapter, an entirely different interaction is proposed, in which the gravitational wave interacts with the plasma and generates fast magneto-acoustic plasma waves, thus dissipating its energy into the plasma. The plasma can then emmit the energy as electromagnetic radiation. Theoretical models for gamma ray burst could be improved a lot if even a small fraction of the GW energy could be converted into EMWs in this fashion. The reason for this is that the energy released in a neutron star binary merger is expected to be released mainly in GWs, whereas the obeserved energy flux is mostly electromagnetic. The dispersion relation derived for this interaction is the most interesting, new, result of this thesis work.

# Contents

# 1 Introduction and overview

A consequence of Einstein's General Theory of Relativity is the existence of the interesting concept of gravitational waves. These waves originate from the most energetic events in our universe, such as colliding neutron star binaries, supernova explosions and gravitational collapses into black holes. They manifest themselves as ripples in space and time, that periodically stretch and compress all present matter and delay and accelerate the time signals from millisecond pulsars, travelling to us over astronomical distances.

## 1.1 Direct gravitational wave detection

The only evidence, until now, of the existence of gravitational waves is the neutron star binary whose orbital radius is slowly decreasing as a consequence of the radiated gravitational energy. The reason that it is so hard to detect gravitational waves (GWs) is that Newton's constant of gravitation is extremely small ($G = 6.7 \cdot 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$). The dimensionless amplitude ($\Delta l/l$) of typical gravitational waves reaching the earth is only of the order of $10^{-17}$. In other words, a rod of one meter in length will oscillate with an amplitude of a millionth of the radius of a hydrogen atom. Still, several direct GW-detectors are being build at present, or have been proposed for the future, that hope to detect these very small vibrations. The best known of these are the *Laser Interferometer Ground Observatory*, LIGO, and the *Laser Interferometer Space Array*, LISA. As the names suggest, the former is a terrestial observatory and the latter a space base one. Needless to say wha enormous engineering achievements these detectors require to detect the oscillations mentioned above.

Another approach to detect GWs is to measure 'relative time displacements' ($\Delta t/t$), instead of position displacements. Time signals from millisecond pulsars can be measured to such precision that time delays caused by the gravitational waves can in principle be measured with the same precision as position displacements as soon as enough stable pulsars can be observed for a sufficiently long time. Dr. A. Lommen et. al. (Berkeley, U.S.A.) plan to use this fact to determine, from GWs, whether the central object of our galaxy is a black hole binary.

## 1.2 Watching gravitational waves

The subject of the first part of this thesis is an entirely different way to *indirectly observe gravitational waves*. It appears that GWs can be converted to electromagnetic waves, viz light, that are much easier to detect.

According to General Relativity, all forms of energy are equivalent, which is one way to formulate the Equivalence Principle. This is reflected by Einstein field equations, which are the equivalent of the non-relativistic Poisson equation for the gravitational potential. Just as the latter gives the gravitational potential as a function of the matter density, the former gives the space-time curvature due to the total energy-momentum density. Gravitational waves are the first order, harmonic solutions for these field equations.

As a result of the Equivalence Principle, not only oscillating matter sources can produce GWs, but also oscillating EM energy densities. In linearized theory, this means that also the reverse process might occur: electromagnetic waves generated by gravitational waves. It will be shown in Sec. 3 that a gravitational wave passing through a constant electromagnetic background field is partially converted into light. In other words, one could indirectly 'watch' gravitational waves in the form of light.

In Sec. 4 an exact calculation will be presented for a linearly polarized EMW passing through an arbitrary EM background field which converts it to a GW which is then reconverted into an EMW. From this calculation it is apparent that such conversions are absolutely inefficient in any laboratory experiment. From an observational view point, however, they could provide a means to indirectly observe the GWs from more energetic, astrophysical phenomena such as supernovæ or merging neutronstar binaries.

## 1.3 Cosmic background radiation

As a second phenomenon where GW to EMW conversions could play an important rôle, the observed fluctuations in the cosmic background radiation are investigated. According to the standard model big bang theories, 20.000 years after the Big Bang, the electron mist that until then obscured the universe, evaporated, and the universe became transparent to the $\approx 2700$ Kelvin thermal radiation.

This radiation is observed as a very isotropic flux of photons, coming to us from all directions. What has not been explained properly are the small fluctuations that do appear in the radiation. If, however, a primordial magnetic field existed in the early universe, gravitational waves travelling through this field could have been converted into electromagnetic waves causing small fluctuations in the otherwise homogeneous radiation. An estimate of this effect is given in Sec. 8.2.

## 1.4 Gamma ray bursts

A final GW to EMW conversion is discussed in the last part of this thesis. In a strongly magnetized plasma surrounding a source of GWs, these waves might generate plasma waves, which in turn produce light in the form of cyclotron, synchrotron etc. radiation. In this case, the interaction is caused by the fact that in a very strong magnetic field, the plasma electrons are frozen to the field lines. The GWs cause displacements of the electrons, which thus induce oscillations in the magnetic field.

The relation to gamma ray bursts is that in these bursts, most of the energy is expected to be released in the form of gravitational waves, whereas the observed energy flux is electromagnetic. A way to solve this problem would be to dissipate a small fraction of the gravitational wave energy into the plasma, which can then radiate this energy as the observed electromagnetic waves. This is a compelling alternative, for instance, to theories suggesting a neutrino fueled fireball.

## 1.5 Overview

The next section, Sec. 2, is a brief introduction to some of the concepts and equations that are needed in later sections. In Sec. 2.1 gravitational waves are derived as weak field solutions of Einstein's field equations and the non-coordinate tedrad formalism, which will be used for most of the calculations, is explained in Sec. 2.2.

In Secs. 3–4, the efficiency of GW to EMW conversion in a vacuum is considered, first in an estimate following Gertsenstein [15] in Sec. 3 and then, in an exact calculation, for an arbitrary EM background field and arbitrary GW components in Sec. 4. Finally, as an astrophysical application, an order of magnitude calculation is given for GW to EMW conversion close to a quacking supernova remnant neutron star, or magnetar, in Sec. 5.

In Sec. 6 the vacuum is replaced by a thin plasma leading to some dispersion of the generated light (section 6.4) and finally, in Sec. 7, pressure gradients are added and the magnetohydrodynamic approximation is considered. From this the plasma waves are derived, needed to provide a GW dissipation mechanism for gamma ray bursts.

The thesis ends with some conclusions in chapter 8 (and in appendix C second order GW-effects are taken into account resulting in the excitation of longitudinal Alfvén-like waves).

## 1.6 Note on units

Throughout this thesis, Gaussian CGS units are used in all equations. In most exact calculations the speed of light and Newton's constant are suppressed by choosing $c = G = 1$ but in numerical estimates, these constants will sometimes reappear to emphasize their importance.

Furthermore, in the first two chapters greek indices will conventionally indicate four-vector components, $\mu = 0, 1, 2, 3$, with time as the zeroth component, and latin indices the spatial components $i, j = 1, 2, 3$. In the last chapter, however, the indices are used the other way around, following the particular convention used by the authors of the articles discussed in that chapter ([10]-[9] and [27]-[21]).

Finally, the signature of flat-space metric is chosen as:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

# 2 General relativity

## 2.1 Gravitational waves

### 2.1.1 Introduction

Gravitational waves, just like any other type of waves, are defined as propagating perturbations of some flat background. Just as water waves are considered to be small ripples on an otherwise flat ocean, GWs are identified as small ripples rolling across spacetime. And just as one neglects the deviation from flatness of the ocean caused by the curved surface of the earth, the tidal forces caused by the gravitational attraction of the sun and the moon, Coriolis forces due to the earths rotation etc., so one ignores the large-scale curved structure of space-time caused by the matter distribution or for instance the presence of a primordial magnetic background field. The GWs originating from supernovæ, explosions in the galactic center, or rotating binary stars can then indeed be described as small ripples on a flat background, as illustrated by the cartoon picture from LISA, Figure 1, showing the gravitational waves of a binary.

As a result of all this, the (linearized) wave equations are easy to derive, as will be done in this section. But one has to keep in mind that these equations are defined only locally, and that they have no meaning globally, where the large scale space-time structure of the universe comes into play.

### 2.1.2 Linearized theory in vacuum

In the General Theory of Relativity, Poisson's equation of gravity is replaced by the equivalent, but covariant field equations, introduced by Einstein:

$$\Box\Phi = 4\pi\rho \qquad \text{POISSON} \tag{2}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \text{EINSTEIN} \tag{3}$$

$$\text{or} \qquad R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{4}$$

The conceptual meaning of these equations is that a localized density distribution curves the space around it and as a result of this, the geodesics for light or particles are deflected in the direction of the center of mass, thus reproducing the effect of a gravitation force.

To derive the gravitational wave equations, the deviations from the Lorentz metric ($\eta_{\mu\nu}$) are assumed to be small. In other words, the weak-field solutions of Einstein's field equations (4) are obtained by assuming:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \qquad |h_{\mu\nu}| \ll 1 \\ h_\mu^{\ \lambda} &= \eta^{\lambda\alpha}h_{\mu\alpha} \qquad \text{and} \qquad h \equiv h_\alpha^{\ \alpha} = \eta^{\sigma\lambda}h_{\sigma\lambda} \end{aligned} \tag{5}$$

which is called the linearized theory of gravity. In this metric the Ricci tensor to first order in $h_{\mu\nu}$ is (see for instance [18]):

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^{\beta}_{\ \mu\nu,\beta} - \Gamma^{\beta}_{\ \mu\beta,\nu} + \Gamma^{\beta}_{\ \mu\nu}\Gamma^{\alpha}_{\ \beta\alpha} - \Gamma^{\alpha}_{\ \mu\beta}\Gamma^{\beta}_{\ \nu\alpha} \\
&= \Gamma^{\beta}_{\ \mu\nu,\beta} - \Gamma^{\beta}_{\ \mu\beta,\nu} \\
&= -\frac{1}{2}h_{\mu\nu,\ \alpha}^{\ \ \ \ \alpha} - \frac{1}{2}(h_{,\mu\nu} - h_{\mu\ \ ,\nu\beta}^{\ \beta} - h^{\beta}_{\ \nu,\ \mu\beta}) \\
&= -\frac{1}{2}h_{\mu\nu,\ \alpha}^{\ \ \ \ \alpha} - \frac{1}{2}\left((\frac{1}{2}\eta_{\mu}^{\ \beta}h - h_{\mu}^{\ \beta})_{,\beta\nu} + (\frac{1}{2}\eta_{\nu}^{\ \beta}h - h_{\nu}^{\ \beta})_{,\beta\mu}\right) \\
&= -\frac{1}{2}h_{\mu\nu,\ \alpha}^{\ \ \ \ \alpha}
\end{aligned} \tag{6}$$

where in the last line, the coordinate conditions are chosen such that $(h_{\mu}^{\ \beta} - \frac{1}{2}\eta_{\mu}^{\ \beta}h)_{,\beta} = 0$ (i.e. a divergence free and traceless solution). The field equations are now given (to first order) by:

$$\begin{aligned}
-\frac{1}{2}h_{\mu\nu,\ \sigma}^{\ \ \ \ \sigma} + \frac{1}{4}\eta_{\mu\nu}h_{,\sigma}^{\ \sigma} &= \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \text{or} \\
\Box\phi_{\mu}^{\ \nu} &= -\frac{16\pi G}{c^4}T_{\mu}^{\ \nu} = -2G^{(1)\ \nu}_{\ \ \mu}
\end{aligned} \tag{7}$$

where $\phi_{\mu}^{\ \nu}$ is defined by:

$$\phi_{\mu}^{\ \nu} = h_{\mu}^{\ \nu} - \frac{1}{2}\eta_{\mu}^{\ \nu}h \qquad \text{with} \qquad \phi_{\mu\ \ ,\nu}^{\ \nu} = 0 \tag{8}$$

The general solutions of (7) are, as mentioned in the previous section, just the solutions of the Poisson equation:

$$\phi_{\mu}^{\ \nu}(r,t) = \frac{4G}{c^4}\int \frac{(T_{\mu}^{\ \nu})_{\text{ret}}d^3x'}{|\vec{r} - \vec{r'}|} \tag{9}$$

### 2.1.3 Plane wave solutions

In a vacuum and the absence of EM fields, where $T^{\mu\nu} = 0$, the field equations (7) reduces to $h_{\mu}^{\ \nu} - \frac{1}{2}\eta_{\mu}^{\ \nu}h = 0$ with as simplest solutions plane waves:

$$\begin{aligned}
\phi_{\mu\nu} &= \Re\left[A_{\mu\nu}\mathrm{e}^{ik_\alpha x^\alpha}\right] \qquad \text{with} \\
k_\alpha k^\alpha &= 0 \qquad \text{NULL VECTOR} \\
A_{\mu\nu}k^\nu &= 0 \qquad \text{TRANSVERSE}
\end{aligned} \tag{10}$$

These solutions are transverse plane waves propagating with the speed of light ($\omega = |\mathbf{k}|$) in the direction $\mathbf{k}/\omega$.

The amplitude of this wave still seems to have six independent components (ten components because $A_{\mu\nu}$ is symmetric (since $T^{\mu\nu}$ is) of which four are fixed by the restriction $A_{\mu\nu}k^\nu = 0$). This is due to a gauge freedom under infinitesimal coordinate

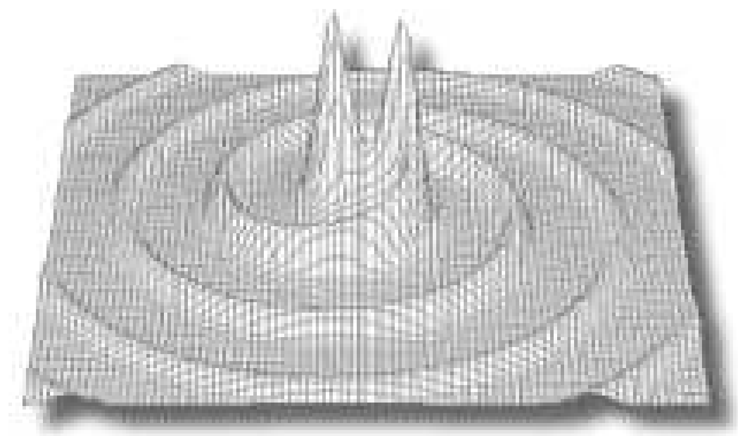

Figure 1: Gravitational wave from binary.

transformations $x'^{\mu} \to x^{\mu} + \xi^{\mu}$ satisfying $\Box\xi = 0$. In this case $\xi^{\mu} = -iC^{\mu} \exp ik_{\alpha}x^{\alpha}$ generates gauge transformations that can arbitrarily alter four of $A_{\mu\nu}$'s components. therefore one has to choose not only a certain coordinate condition (as in (6)) but also a specific gauge.

### 2.1.4 Transverse traceless gauge

Defining the observer velocity by $u^{\nu}$, one can perform a gauge transformation such that $A_{\mu\nu}u^{\nu} = 0$ which fixes three of the components, and another transformation to set $A^{\mu}{}_{\mu} = 0$ wich fixes the last one. What remains are the two dynamical degrees of freedom of the gravitational field (the two polarizations). Summarizing:

$$A_{\mu\nu}u^{\nu} = A_{\mu\nu}k^{\nu} = A^{\mu}{}_{\mu} = 0 \tag{11}$$

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{+} & A_{\times} & 0 \\ 0 & A_{\times} & -A_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

One result of this gauge is that $\phi_{\mu\nu} = h_{\mu\nu}$ which makes the analogy with Poisson's equation complete.

### 2.1.5 Polarization

Just like electromagnetic waves, all gravitational waves can be decomposed into two *linearly polarized* components, or in two *circularly polarized* components that give the geodesic deviation in a certain direction ([18]). Linearly polarized, electromagnetic waves with polarization vectors $\mathbf{e}_x$ or $\mathbf{e}_y$, travelling in the $z$-direction, cause a test particle to oscillates in the $x$- or $y$-direction respectively (with respect to an inertial frame).

For a gravitational wave, from (12), the *unit linear-polarization tensors* are, by analogy (see for instance [19]):

$$
\begin{aligned}
e_{+}^{\mu\nu} &\equiv \frac{1}{\sqrt{2}}(e_x^{\mu}e_x^{\nu} - e_y^{\mu}e_y^{\nu}) \\
e_{\times}^{\mu\nu} &\equiv \frac{1}{\sqrt{2}}(e_x^{\mu}e_y^{\nu} + e_x^{\nu}e_y^{\mu})
\end{aligned} \tag{13}
$$

The effect of such a polarized wave on a ring of test particles around a central particle is as follows: in the plane transverse to the GW-propagation,[1] the ring is deformed into an ellipse that pulsates in and out along the $x$- and $y$-axes ($\mathbf{e}_+$) or at an angle of 45° with respect to these axes ($\mathbf{e}_\times$). In Sec. 6, plane $+$-polarized GWs will be considered, entering a thin plasma.

*Unit circular-polarization tensors* for gravitational waves are again constructed by analogy to electromagnetic polarization vectors, $\mathbf{e}_R = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ and $\mathbf{e}_L = (\mathbf{e}_x - i\mathbf{e}_y)/\sqrt{2}$:

$$
\begin{aligned}
\mathbf{e}_R &= \frac{1}{\sqrt{2}}(\mathbf{e}_+ + i\mathbf{e}_\times) \\
\mathbf{e}_L &= \frac{1}{\sqrt{2}}(\mathbf{e}_+ - i\mathbf{e}_\times)
\end{aligned} \tag{14}
$$

One can see that $\mathbf{e}_R$ and $\mathbf{e}_L$ rotate a deformation of a ring of test particles in the counterclockwise and the clockwise direction respectively.

An interesting characteristic of these polarizations is that the gravitational wave is invariant under a $\pi$ rotation, which reflects the quantum behaviour of the associated massless *graviton* particles.

The classical radiation field of a spin $S$ particle is invariant under a $2\pi/S$ rotation (for instance $2\pi$ for spin-1 photons and $4\pi$ for spin-$\frac{1}{2}$ neutrinos). Apparently, gravitons are spin-2 particles. This also follows from the fact that the two orthogonal linear polarizations of a spin-$S$ field are inclined to each other at an angle $\frac{1}{2}\pi/S$. Indeed $\mathbf{e}_+$ and $\mathbf{e}_\times$ are inclined to each other at $\frac{1}{4}\pi$ (and for instance for photons $\mathbf{e}_x$ and $\mathbf{e}_y$ at $\frac{1}{2}\pi$).

### 2.1.6 Stress-energy carried by GW

A very important consequence of Einstein's equivalence principle is that *local gravitational energy-momentum*, or its density, is not uniquely defined. Non-vanishing stress-energy implies, after all, curved space, whereas the equivalence principle always allows one to choose a locally inertial (flat) coordinate system. Gravitational energy is therefore non-localizable. One has to average over several wavelengths to arrive at a well defined energy content in a certain region and talk about an *effective* smeared-out stress-energy of gravitational waves.

In an arbitrary gauge, this effective stress-energy for a GW follows from contracting the weak field equations (4) with $h^{\mu}{}_{\nu,\rho}$ leading to (see [35] p. 94 or [18] p. 955):

[1] in the reference frame of the central particle

$$T^{\text{EFF}}_{\mu\nu} = \frac{c^4}{16\pi G}\langle h_{\alpha\beta,\mu}h^{\alpha\beta}_{\ \ ,\nu} - \frac{1}{2}h_{,\mu}h_{\nu} - h^{\alpha\beta}_{\ \ \beta}h_{\alpha\mu,\nu} - h^{\alpha\beta}_{\ \ \beta}h_{\alpha\nu,\mu}\rangle \tag{15}$$

which in the transverse-traceless gauge discussed before, reduces to:

$$T^{\text{EFF}}_{\mu\nu} = \frac{c^4}{16\pi G}\langle h^{TT}_{ij,\mu}h^{TT}_{ij,\nu}\rangle \tag{16}$$

This is the expression that will be used in estimating the amount of gravitational energy that can be converted into electromagnetic radiation or vice versa when a GW travels through a strong electromagnetic background field in for instance (27) in Sec. 3.

For a plane, linearly polarized GW of the form $h_{\mu\nu} = \Re\left\{(A_+ e_{+\mu\nu} + A_\times e_{\times\mu\nu})\mathrm{e}^{ik(z-t)}\right\}$, the only non-vanishing components are:

$$T^{\text{EFF}}_{tt} = T^{\text{EFF}}_{zz} = -T^{\text{EFF}}_{tz} = -T^{\text{EFF}}_{tz} = \frac{k^2 c^4}{16\pi G}(|A_+|^2 + |A_\times|^2) \tag{17}$$

## 2.2 Tedrad formalism

In describing physics on curved space-times, it is customary to work in the socalled tedrad formalism. This formalism is usually the most efficient in computing curvature and many other quantities. Also it provides the closest connection to measurement. After all, a measurement made by a (for instance accelerated) observer is always made in its locally inertial reference frame. For this reason, most calculations in this thesis will, at some point, involve the introduction of a certain tedrad system to describe what physics is going on. This section provides a short overview of the concepts that will be used in the following sections.

### 2.2.1 Spinors and cosmological models

In the General Theory of Relativity, the only equations of physical interest are covariant tensor equations. Half integer spin particles, however, do not transform like tensors, but like spinors, so the general 'recipe' of replacing all physical quantities by tensors, and using covariant derivatives everywhere instead of partial derivatives, does not work any more as soon as electrons or other spin $\frac{1}{2}$ paricles are involved. It is therefore not possible to describe the effect of gravitation on the fields of electrons and most other charged particles (see [36]). Using non-coordinate methods, however, allows one to incorporate spinors into calculations on the same footing as tensors.

Another area where the tedrad formalism is of great importance is in the study of cosmological models. A lot of work on this is done by G. F. R. Ellis ([12]) and H. van Elst (in [13]). They set up these models in a $1+3$ covariant description where all tensor components are projected onto the average velocity vector and the directions orthogonal to it. The six propagation equations and six constraint equations resulting from this can only be completed by putting them into tedrad form, where one then has additional Ricci and Jacobi identities for the basis vectors.[2]

Since the problems that will be discussed in this thesis involve a electron plasma in a cosmological background, it seems natural to adopt the tedrad formalism to describe these phenomena.

### 2.2.2 Decomposition to locally inertial frame

From the equivalence principle one is always free to choose a locally inertial coordinate system at every point $P$ (local metric $\eta_{\mu\nu}$), which is related to a general non-inertial system, $x^\mu$ by:

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\frac{\partial \xi_P^\alpha(x)}{\partial x^\mu}\right)_{x=P} \left(\frac{\partial \xi_P^\beta(x)}{\partial x^\nu}\right)_{x=P} \eta_{\alpha\beta} \qquad (18) \\ &= \lambda^\alpha{}_\mu \lambda^\beta{}_\nu \eta_{\alpha\beta} \end{aligned}$$

[2] The tedrad form also allows the set of equations to be put into a symmetric hyperbolic normal form which is easier to solve.

One can easily see any other choice of general coordinates $x'^{\mu}$ would have led to the same expression, with:

$$\lambda'^{\alpha}_{\ \mu} = \frac{\partial \xi^{\alpha}_{P}}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \xi^{\alpha}_{P}}{\partial x^{\nu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \lambda^{\alpha}_{\ \nu} \tag{19}$$

Apparently, $\lambda^{\alpha}_{\ \mu}$ does not transform like a single tensor, but as four covariant vector fields, hence the name *tedrad* or *vierbein*. Using this tedrad, any contravariant vector field or tensor can be decomposed along the locally inertial coordinate axes:

$$\begin{aligned} F^{(\mu)} &\equiv \lambda^{(\mu)}_{\ \nu} F^{\nu} \\ G^{(\mu\nu)} &\equiv \lambda^{(\mu)}_{\ \alpha} \lambda^{(\nu)}_{\ \beta} G^{\alpha\beta} \end{aligned} \tag{20}$$

where the brackets denote which tedrad vector is contracted, and illustrate that by contracting the tensor fields with the tedrad vectors, they are decomposed into a set of scalars.[3] This is precisely the reason why the treatment of spinors (of for instance Dirac electrons) is no longer different from that of tensors. Both can be put into an action as scalars.

In a more informal way of speaking, what is meant by the tedrad decomposition, is that at each point in space, physical quantities are projected onto artificial, orthogonal axis that 'stick out' of the curved space as it where. Hence the name *non-coordinate* frames.

### 2.2.3 General covariance and Lorentz invariance

To construct an action from the decomposed tensors, both general covariance and Lorentz invariance of the action must be satisfied. The scalar components are, after all, defined with respect to an arbitrarily chosen ONF, so the action should be invariant under a redefinition of this frame, viz under Lorentz transformations.

$$\begin{aligned} F^{(\mu)}(x) &= \Lambda^{\mu}_{\ \nu}(x) F^{(\nu)}(x) \\ G^{(\mu\nu)} &= \Lambda^{\mu}_{\ \alpha}(x) \Lambda^{\nu}_{\ \beta}(x) G^{(\alpha\beta)}(x) \end{aligned} \tag{21}$$

The same conditions apply to the tedrad vectors themselves.

Now, to construct an appropriate action, one does not only need the fields, decomposed as scalars, but also derivatives of the fields. Since al the fields are decomposed as scalars, the derivatives also have to be scalars. At the same time though they have to be invariant under Lorentz transformations. The derivative that satisfies these conditions is (see [13], [33] or [36]) :

$$\nabla_{\mu} = \lambda^{\ \nu}_{(\mu)} \frac{\partial}{\partial x^{\nu}} - \lambda^{(\alpha)}_{\ \beta} \lambda^{\ \gamma}_{(\mu)} \frac{\partial}{\partial x^{\gamma}} \lambda^{\ \beta}_{(\delta)} \tag{22}$$

[3]Later on these brackets will be omitted for brevity, assuming that it is clear to the reader what the different indices mean.

**Summarizing:** just as one obtains General Relativity by replacing the special relativistic action and field equation by tensor equations, and the partial derivatives by covariant deriveves, one obtains a coordinate free description of General Relativity by decomposing all tensors or spinors into scalars (except for the tedrad vectors itself), and replacing all the partial derivatives with the covariant and Lorentz invariant derivative (22).

### 2.2.4 The connection

In a particular set of coordinates, space-time is described by a metric given by $g^{\mu\nu}(x)$ and a connection, that supplies the differential properties of space-time: the Christoffel symbols. In the non-coordinate formalism, space-time is is locally flat (metric $\eta^{\mu\nu}$) with respect to a particular set of tedrad vectors, and its differential properties are given by a connection in the form of the socalled *Ricci rotation coefficients*.

The specific form of these coefficients follows readily from (22).

$$\begin{aligned} \nabla_{\lambda_\beta}\lambda_\alpha &= \Gamma^{\gamma}{}_{\alpha\beta}\lambda_\gamma \qquad \Leftrightarrow \\ \Gamma^{\alpha}{}_{\beta\mu} &= \lambda^{(\alpha)}_{\ \beta}\lambda_{(\mu)}^{\ \gamma}\frac{\partial}{\partial x^\gamma}\lambda_{(\delta)}^{\ \beta} \end{aligned} \tag{23}$$

In tedrad components, covariant derivatives can thus be calculated completely analogous to the tensorial form:

$$\nabla_\alpha T_{\beta\gamma} = \lambda_{(\alpha)}^{\ \nu}\frac{\partial T_{\beta\gamma}}{\partial x^\nu} - \Gamma^{\delta}{}_{\beta\alpha}T_{\delta\gamma} - \Gamma^{\delta}{}_{\gamma\alpha}T_{\beta\delta} \tag{24}$$

This derivative will be used in all the plasma calculations is Sec. 6.

## 3 Estimate of interaction efficiency

According to General Relativity, gravitational waves and electromagnetic waves propagate with the same speed and, in vacuum, obey the same dispersion relation. If there is a linear dependence between the waves, they can therefore resonate and transfer energy ([15]). In this section propagation of light ($F^{\mu\nu}$) in the presence of a strong magnetic background field ($F^{(0)\mu\nu}$) that is constant in space-time, is considered.

The stress-energy tensor for the combined field ($F'^{\mu\nu} = F^{(0)\mu\nu} + F^{\mu\nu}$) consists of:

1. the square of the constant field term that does not generate gravitational waves,
2. the square of the field of the electromagnetic wave which does not create GW's either, and
3. the interference term which is proportional to the background field and produces an oscillating source term for gravitational waves.

The field equations for the resonant term, which is the only relevant part of the stress-energy tensor, are given by:

$$\begin{aligned} \Box\phi^{\mu}{}_{\nu} &= -\frac{8G}{c^4}\left[F^{(0)\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\eta^{\mu}{}_{\nu}F^{(0)\alpha\beta}F_{\alpha\beta}\right] \\ &= -\frac{8G}{c^4}F^{(0)\mu\alpha}F_{\nu\alpha} \end{aligned} \tag{25}$$

where the second line results from choosing certain convenient coordinates, for instance such that:

$$\begin{aligned} F^{(0)\alpha\beta}F_{\alpha\beta} &\propto (\mathbf{E}\cdot\mathbf{E}^{(0)} - \mathbf{B}\cdot\mathbf{B}^{(0)}) \qquad \text{with} \\ \mathbf{E}^{(0)} &= 0 \qquad \text{and} \qquad \mathbf{B}\perp\mathbf{B}^{(0)} \end{aligned} \tag{26}$$

For an incoming lightwave along the $x$-axis (wavevector $|\vec{k}| = k_x = \omega/c$), with no absorption or scattering ($b(x) = b$), try plane wave solutions with amplitudes normalized to unit energy density (16) and $k^{\alpha}_{\text{GW}} = k^{\alpha}_{\text{EMW}} = k^{\alpha}$, e.g. gravitational waves produced along the $x$-axis.

$$\begin{aligned} t_{00} &\sim \frac{c^4}{16\pi G} < (h_{\mu\nu,0})^2 >= \frac{c^4}{16\pi G}\langle\phi_{\mu\nu,0}\rangle^2 \\ T_{00} &= \frac{1}{4\pi}(\mathbf{E}^2 + \mathbf{B}^2) \end{aligned} \tag{27}$$

except for the overall 'conversion factors' $a(x)$ and $b$:

$$\begin{aligned} F_{\mu\nu} &= \Re\left[b\sqrt{4\pi}f_{\mu\nu}\mathrm{e}^{ik_\alpha x^\alpha}\right] \quad , \quad f^{0\nu}f_{0\nu} = 1 \quad \text{and} \quad f^{ij}f_{ij} = 1 \\ \phi^{\mu\nu} &= \Re\left[a(x)\sqrt{\frac{16\pi G}{c^4k^2}}\zeta^{\mu\nu}\mathrm{e}^{ik_\alpha x^\alpha}\right] \quad , \quad \zeta^{\mu\nu}\zeta_{\mu\nu} = 1 \quad , \quad \zeta^{\mu}{}_{\mu} = 0 \end{aligned} \tag{28}$$

For slowly varying amplitudes, neglecting terms quadratic in $d/dx$ and of course $k_\mu k^\mu = 0$, the field equations reduce to:

$$\begin{aligned}\Box\phi^\mu{}_\nu &= \sqrt{\frac{16\pi G}{c^4}}\frac{\zeta^\mu_\nu}{k_x}\left(k_\alpha k^\alpha a(x) + \frac{d^2a(x)}{dx^2} + 2ik_x\frac{da(x)}{dx}\right)\mathrm{e}^{ik_\beta x^\beta} \\ &= -\frac{16\sqrt{\pi}bG}{c^4}F^{(0)\mu\alpha}f_{\nu\alpha}\mathrm{e}^{ik_\beta x^\beta}\end{aligned} \tag{29}$$

which is a differential equation that can be solved for $a(x)$:

$$\frac{da(x)}{dx} = ib\sqrt{\frac{4G}{c^4}}F^{(0)\mu\alpha}f_{\nu\alpha}\zeta_\mu{}^\nu \tag{30}$$

$$a(x) = ib\sqrt{\frac{4G}{c^4}}f_{\nu\alpha}\zeta_\mu{}^\nu\int_0^x F^{(0)\mu\alpha}(x')dx' + a(0) \tag{31}$$

One can already see from this that the energy of the generated GW grows with distance ($a^*a \uparrow x$). In the figures below, the polarization of the generated GW is shown for two different polarizations for the incident EMW. In Figure 2, the magnetic wave component is orthogonal to the magnetic background field, as in (26): $\mathbf{B}_{\text{EM}}\perp\mathbf{B}^{(0)}\perp\mathbf{k}_{\text{EM}}$, and the resulting GW is $\times$-polarized.

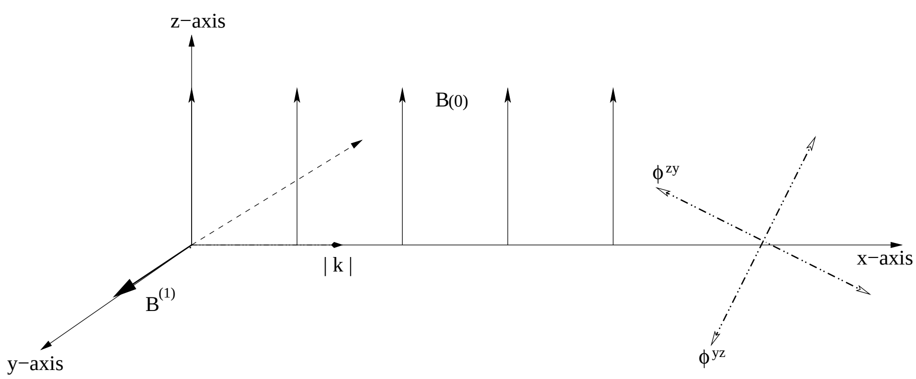


Figure 2: Orthogonal magnetic fields.

One could of course also choose the incoming EMW to have a magnetic component parallel to the background field. A similar calculation shows that for $\mathbf{B}_{\text{EM}} \| \mathbf{B}^{(0)} \perp \mathbf{k}_{\text{EM}}$, the resulting GW is +-polarized. This is depicted in Figure 3.

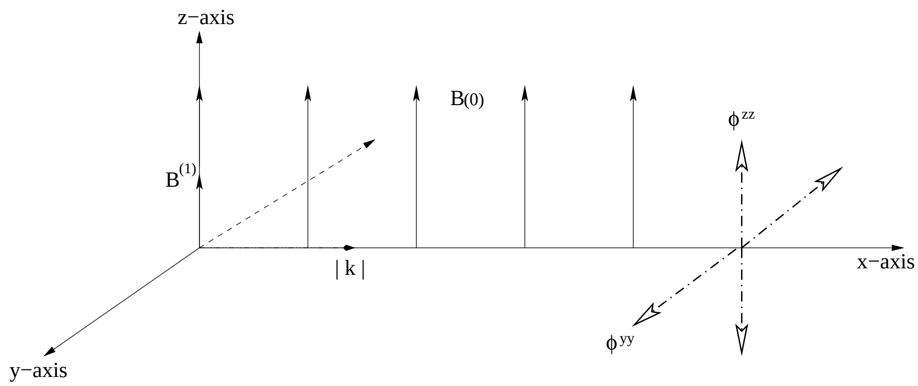


Figure 3: Parallel magnetic fields.

Linear combinations of these electromagnetic configurations lead to circularly polarized GWs (or any other polarization).

**Efficiency.** To calculate the efficiency, assume:

- a static, homogeneous magnetic background field, $F^{(0)}$,
- a typical length- or timescale for the interaction region, $L = Tc$,
- unit convolutions of dimensionless amplitudes,
- and no incoming gravitational waves, $a(0) = 0$.

The energy transfer efficiency from EMWs to GWs is then given by:

$$\alpha = \left\| \frac{a(x)}{b} \right\|^2 = \frac{4G}{c^4} F^{(0)2} L^2 \tag{32}$$

This efficiency is the same for EMW $\Rightarrow$ GW and GW $\Rightarrow$ EMW conversions since the relations are symmetric under time reversal. For a neutronstar binary or magnetar with a large surface magnetic field $F^{(0)} \approx 10^{16}$ Gauss and an interaction region (where one can speak of plane GWs) from $R_1 = 180$ km to $R_2 = 500$ km, and a dipolar decay of the magnetic field, something of the order of $10^{-8}$ of the energy could be converted, which still might be substantial considering the huge amounts of energy released in supernovæ and binary mergers.

**Fluctuations in the CBR.** A very interesting example of GW to EMW conversions, could have occured in the early universe. The observed fluctuations of in the cosmic background radiation might be explained by gravitational waves travelling throught the magnetic field that, according to some cosmological models existed throughout space at that time. If a fraction of the GW energy could be converted into EMWs, this could explain the $\sim 10^{-5}$ relative anomalies in the GBR. This will be discussed in some more detail in Sec. 8.

# 4 Exact calculation in vacuum.

Motivated by the result of the preceding crude estimate, this section proceeds to the exact, analytical derivation of a conversion of photons into gravitons and back into photons again.

## 4.1 Photons to gravitons

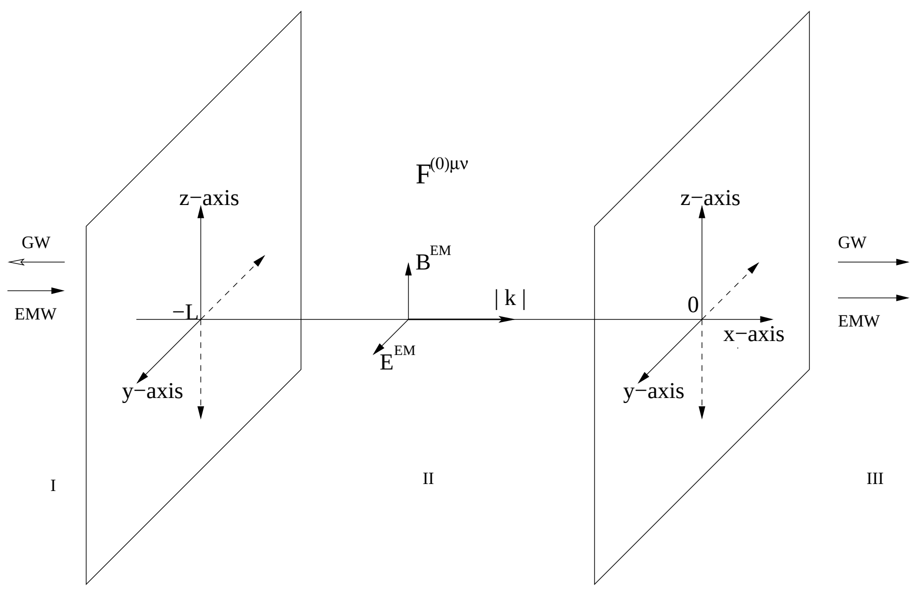


Figure 4: Photon to graviton conversion.

Consider the arrangement shown in Figure 4 ([26] and [7]), where an electromagnetic wave is partially converted into a gravitational wave. The coordinates are again chosen such that the incident electromagnetic wave propagates along the positive $x$-axes. Furthermore, the waves are taken to be linearly polarized plane waves: $B_z = E_y = A \exp ik(x - ct)$. For a completely general electromagnetic background field the combined Faraday tensor is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y + A\mathrm{e}^{ik(x-ct)} & E_z \\ -E_x & 0 & B_z + A\mathrm{e}^{ik(x-ct)} & -B_y \\ -E_y - A\mathrm{e}^{ik(x-ct)} & -B_z - A\mathrm{e}^{ik(x-ct)} & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{33}$$

As in Sec. 3, the stress-energy tensor consists of two squared terms, that do not produce gravitational waves, and the resonant term of interest, which in this case is:

$$T_\mu{}^\nu = \frac{A}{4\pi}\begin{pmatrix} -(E_y+B_z) & -(E_y+B_z) & E_x & B_x \\ (E_y+B_z) & (B_z+E_y) & E_x & B_x \\ -E_x & -E_x & (B_z-E_y) & -(B_y+E_z) \\ -B_x & -B_x & -(B_y+E_z) & -(B_z-E_y) \end{pmatrix} \mathrm{e}^{ik(x-ct)} \quad (34)$$

satisfying the usual conservation condition $T_\mu{}^\nu{}_{;\nu}=0$. So for the ten independent components of the stress-energy tensor, the Einstein equations (7) read:

$$\begin{aligned} \Box\phi_{I\mu}{}^\nu &= 0 \\ \Box\phi_{II\mu}{}^\nu &= \zeta_\mu{}^\nu \mathrm{e}^{ik(x-ct)} \\ \Box\phi_{III\mu}{}^\nu &= 0 \end{aligned} \quad (35)$$

where the subscripts $I$, $II$ and $III$ refer to the regions as indicated in Figure 4. The amplitude $\zeta_\mu{}^\nu$ is defined by:

$$\zeta^{\mu\nu} = -\frac{4AG}{c^4}\begin{pmatrix} -(E_y+B_z) & -(E_y+B_z) & E_x & B_x \\ -(E_y+B_z) & (E_y+B_z) & -E_x & -B_x \\ E_x & -E_x & (B_z-E_y) & -(B_y+E_z) \\ B_x & -B_x & -(B_y+E_z) & -(B_z-E_y) \end{pmatrix} \quad (36)$$

The general plane wave solutions satisfying (35) are:

$$\phi_{I\mu}{}^\nu = \zeta_\mu{}^\nu A_{(\mu\nu)}\mathrm{e}^{-ik(x+ct)} \quad (37)$$

$$\phi_{II\mu}{}^\nu = \zeta_\mu{}^\nu \left(\frac{x+L}{2ik}\mathrm{e}^{ik(x-ct)} + B_{(\mu\nu)}\mathrm{e}^{ik(x-ct)} + C_{(\mu\nu)}\mathrm{e}^{-ik(x+ct)}\right) \quad (38)$$

$$\phi_{III\mu}{}^\nu = \zeta_\mu{}^\nu D_{(\mu\nu)}\mathrm{e}^{ik(x-ct)} \quad (39)$$

where the arbitrary constants $A_{(\mu\nu)}, B_{(\mu\nu)}, C_{(\mu\nu)}, D_{(\mu\nu)}$ still have to be determined from the coordinate conditions ($\phi_\mu{}^\nu{}_{,\nu}=0$ as in (8)) and the continuity conditions at $x=-L$ and $x=0$:

$$\phi_{I\mu}{}^\nu|_{x=-L} = \phi_{II\mu}{}^\nu|_{x=-L} \quad (40)$$

$$\phi_{II\mu}{}^\nu|_{x=0} = \phi_{III\mu}{}^\nu|_{x=0} \quad \text{and} \quad (41)$$

$$\phi_{I\mu}{}^\nu{}_{,1}|_{x=-L} = \phi_{II\mu}{}^\nu{}_{,1}|_{x=-L} \quad (42)$$

$$\phi_{II\mu}{}^\nu{}_{,1}|_{x=0} = \phi_{III\mu}{}^\nu{}_{,1}|_{x=0} \quad (43)$$

Some tedious algebra results in the following constants:

$$
\begin{aligned}
A_{(\mu\nu)} &= \frac{1}{4k^2}\begin{pmatrix} 1-\mathrm{e}^{2ikL} & \mathrm{e}^{-2ikL}-1 & \mathrm{e}^{-2ikL}-1 & \mathrm{e}^{-2ikL}-1 \\ 1-\mathrm{e}^{2ikL} & \mathrm{e}^{-2ikL}-1 & -\mathrm{e}^{2ikL}+1 & -\mathrm{e}^{2ikL}+1 \\ 1-\mathrm{e}^{2ikL} & \mathrm{e}^{-2ikL}-1 & \mathrm{e}^{-2ikL}-1 & \mathrm{e}^{-2ikL}-1 \\ 1-\mathrm{e}^{2ikL} & \mathrm{e}^{-2ikL}-1 & -\mathrm{e}^{2ikL}+1 & \mathrm{e}^{-2ikL}-1 \end{pmatrix} \qquad (44)\\
B_{(\mu\nu)} &= \frac{1}{4k^2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\\
C_{(\mu\nu)} &= \frac{1}{4k^2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad , \quad D_{(\mu\nu)} = \frac{L}{2ik}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

In these equations the indices in brackets denote the constant factors of the corresponding index positions in $\zeta_\mu^\nu$.

If the length of the device is set to a half-integer multiple of the wavelength, the amplitude of the retreating gravitational waves vanishes (for $L = n\lambda/2 \rightarrow \exp(\pm 2ikL) = 1$). In this case gravitational waves are only generated in region III, which are linearly dependent on the length of the static background field. The complete expression for these waves in terms of the controlled parameters (background field, incident EM-waves and length of the device) becomes:

$$
\begin{aligned}
\phi_\mu{}^\nu &= \frac{L}{2ik}\zeta_\mu{}^\nu \mathrm{e}^{ik(x-ct)} \\
&= \xi_\mu{}^\nu \mathrm{e}^{ik(x-ct)} = h_\mu{}^\nu
\end{aligned} \qquad (45)
$$

The last line follows from: $h_\mu{}^\nu = \phi_\mu{}^\nu - \frac{1}{2}\eta_\mu{}^\nu \phi_\alpha{}^\alpha$ with, from (36), $\phi_\alpha{}^\alpha = 0$. As a result of this, (45) fully describes the metric of the GW entering the next part of the detection device (Sec. 4.2).

#### 4.1.1 Efficiency

From the equation for the metric (45), one can obtain an expression for the energy flux (15) carried by the generated gravitational wave:

$$
\begin{aligned}
W = t^{01} &= \frac{c^2}{16\pi G}\left(\dot{h}_{23}^2 + (\dot{h}_{22}^2 - \dot{h}_{33}^2)^2\right) \\
&= \left(\frac{G}{4\pi c^4}\right)(LA)^2\left((B_y + E_z)^2 + (E_y + B_z)^2\right)
\end{aligned} \qquad (46)
$$

Apparently, there is no coupling when a linearly polarized EMW with components $B_z = E_y$, propagates through a background field with $E_y = -B_z$ and $E_z = -B_y$. Because of

symmetry arguments, the same result will hold for an incident linearly polarized EMW with $B_y = E_z$. For a pure magnetic field (i.e. $F^{\mu\nu} \rightarrow B_z$) this reduces to the estimated relation derived in the previous section, except for an overall factor of $1/16\pi$, whereas for a pure magnetic or electric field aligned with the $x$-axis, the energy flux vanishes! Note the quadratic dependence of the energy flux on both the length of the static field and the amplitude of the incident EM-wave.

## 4.2 Gravitons to photons

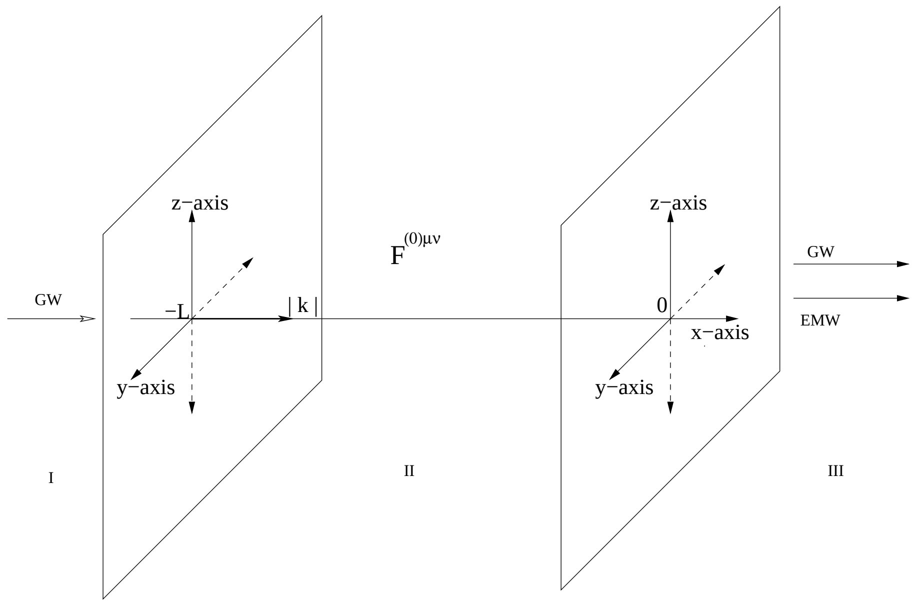


Figure 5: Graviton to photon conversion.

Let us proceed now to the second part of the detection device, where the incident gravitational waves are reconverted to electromagnetic waves (Figure 5).[4] The same symmetry relative to the $x$-axis as before is used to solve the Maxwell equations,

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0 \quad \text{and} \quad (\sqrt{-g}F^{\mu\nu})_{,\nu} = F^{\mu\nu}{}_{,\nu} = 0 \tag{47}$$

in the oscillating metric perturbation $h_{\mu\nu}$. Because of the symmetry, only the derivatives with respect to $x$ and $ct$ contribute (to first order, these are just partial derivatives, since the connection is itself proportional to $h_{\mu\nu}$):

$$\begin{array}{rcl} F_{23,1} = F_{23,0} = 0 & , & F^{01}{}_{,1} = F^{10}{}_{,0} = 0 \\ F_{13,0} + F_{30,1} = 0 & , & F^{31}{}_{,1} + F^{30}{}_{,0} = 0 \\ F_{21,0} + F_{02,1} = 0 & , & F^{21}{}_{,1} + F^{20}{}_{,0} = 0 \end{array} \tag{48}$$

which results in the first field components $F_{23} = F_{23}^{(0)}$ and $F^{01} = E_x^{(0)}$. In the specified metric, the contravariant components for $F^{\mu\nu}$ are derived by:

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} \tag{49}$$

[4] After travelling through the first background field, the remaining light is blocked by a screen that does not influence the propagation of the GWs.

$$
\begin{aligned}
&= (\eta^{\mu\alpha} + h^{\mu\alpha})(\eta^{\nu\beta} + h^{\nu\beta})F_{\alpha\beta} \\
&= (\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\alpha}h^{\nu\beta} + h^{\mu\alpha}\eta^{\nu\beta})F_{\alpha\beta} + \mathcal{O}(h^2)
\end{aligned}
$$

so for $F^{23}$:

$$
F^{23} = B_x^{(0)} + (E_y^{(0)} - B_z^{(0)})h_{30} + (B_y^{(0)} + E_z^{(0)})h_{12} \tag{50}
$$

Similarly, the contravariant components of the other covariant differential equations are given by (using the symmetries in the metric tensor):

$$
\begin{aligned}
F^{21,0} + F^{02,1} &= (E_y^{(0)} - B_z^{(0)})h_{22,1} + (B_y^{(0)} + E_z^{(0)})h_{23,1} \\
F^{13,0} + F^{30,1} &= (E_y^{(0)} - B_z^{(0)})h_{23,1} + (B_y^{(0)} + E_z^{(0)})h_{33,1}
\end{aligned} \tag{51}
$$

which together with (48) leads to the wave equations:

$$
\begin{aligned}
\Box F^{21} &= \\
\Box F^{20} &= (E_y^{(0)} - B_z^{(0)})h_{22,11} + (B_y^{(0)} + E_z^{(0)})h_{23,11} \\
&= \alpha \mathrm{e}^{ik(x-ct)} \\
\Box F^{31} &= \\
\Box F^{30} &= (E_y^{(0)} - B_z^{(0)})h_{23,11} + (B_y^{(0)} + E_z^{(0)})h_{33,11} \\
&= \beta \mathrm{e}^{ik(x-ct)}
\end{aligned} \tag{52}
$$

with,

$$
\begin{aligned}
\alpha &= -k^2 \left\{ (E_y^{(0)} - B_z^{(0)})\xi_{22} + (B_y^{(0)} + E_z^{(0)})\xi_{23} \right\} \\
\beta &= -k^2 \left\{ (E_y^{(0)} - B_z^{(0)})\xi_{23} + (B_y^{(0)} + E_z^{(0)})\xi_{33} \right\}
\end{aligned} \tag{53}
$$

In the same fashion, one can express (51) in $\alpha \mathrm{e}^{ik(x-ct)}/ik$ and $\beta \mathrm{e}^{ik(x-ct)}/ik$. The general solutions of (48), (51) and (52) are just in- and outgoing plane wave solutions plus a linear term expressing how the EMWs grow in amplitude while the GW loses energy. The electromagnetic field components are given by the solutions:

$$
\begin{aligned}
B_z &= B_z^{(0)} + \frac{\alpha x}{2ik}\mathrm{e}^{ik(x-ct)} + A\mathrm{e}^{ik(x-ct)} + B\mathrm{e}^{-ik(x+ct)} \\
-E_y &= -E_y^{(0)} + \left(\frac{\alpha x}{2ik} - \frac{\alpha}{2k^2}\right)\mathrm{e}^{ik(x-ct)} + A\mathrm{e}^{ik(x-ct)} - B\mathrm{e}^{-ik(x+ct)} \\
B_y &= H_y^{(0)} + \frac{\beta x}{2ik}\mathrm{e}^{ik(x-ct)} + C\mathrm{e}^{ik(x-ct)} + D\mathrm{e}^{-ik(x+ct)} \\
-E_z &= -E_z^{(0)} + \left(\frac{\beta x}{2ik} - \frac{\beta}{2k^2}\right)\mathrm{e}^{ik(x-ct)} + C\mathrm{e}^{ik(x-ct)} - D\mathrm{e}^{-ik(x+ct)}
\end{aligned} \tag{54}
$$

where the arbitrary constants $A, B, C$ and $D$ still have to be determined from continuity conditions on the oscillating field ($i = x, y, z$):

$$\begin{aligned} E_{Ii}|_{x=-L} = E_{IIi}|_{x=-L} \qquad &\text{and} \qquad B_{Ii}|_{x=-L} = B_{IIi}|_{x=-L} \\ E_{IIi}|_{x=0} = E_{IIIi}|_{x=0} \qquad &\text{and} \qquad B_{IIi}|_{x=0} = B_{IIIi}|_{x=0} \end{aligned} \tag{55}$$

### 4.2.1 Tedrad system

First, however, one has to realize that a measurement of the field components in this curved metric only makes sense in a local Cartesian coordinate frame (ONF) on the world line of an observer, as was discussed in Sec. 2.2. It is therefore customary to employ a non-coordinate frame for an observer at rest with respect to the GW, the tedrad $\lambda^{\mu}_{(\alpha)}$, ([6], [36], [24] and [32]). For these tedrad vectors to form a orthonormal frame, they have to obey the orthonormality condition:

$$g_{\mu\nu}\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)} = \eta_{(\alpha\beta)} \tag{56}$$

In physical terms, this is nothing more than a local coordinate transformation to an inertial system.

In this case, the first components of $\lambda^{\mu}_{(2)}$ and $\lambda^{\mu}_{(3)}$ can be chosen to be zero, and the last component of $\lambda^{\mu}_{(2)}$ to be zero to remove the arbitrarity of a rotation around the $x$-axis. The other components then follow from (56).

$$\begin{aligned} \lambda^{\mu}_{(0)} &= (1 + \frac{1}{2}h_{00}, 0, 0, 0) \quad , \quad \lambda_{(0)\mu} = (\frac{1}{2}h_{00} - 1, h_{10}, h_{20}, h_{30}) \\ \lambda^{\mu}_{(1)} &= (h_{10}, 1 - \frac{1}{2}h_{11}, -h_{12}, -h_{13}) \quad , \quad \lambda_{(1)\mu} = (0, 1 + \frac{1}{2}h_{11}, 0, 0) \\ \lambda^{\mu}_{(2)} &= (h_{20}, 0, 1 - \frac{1}{2}h_{22}, 0) \quad , \quad \lambda_{(2)\mu} = (0, h_{12}, 1 + \frac{1}{2}h_{22}, h_{23}) \\ \lambda^{\mu}_{(3)} &= (h_{30}, 0, -h_{23}, 1 - \frac{1}{2}h_{33}) \quad , \quad \lambda_{(3)\mu} = (0, h_{13}, 0, 1 + \frac{1}{2}h_{33}) \end{aligned} \tag{57}$$

The decomposition of the electromagnetic tensor in this ONF is:

$$F_{(\alpha\beta)} = F^{\mu\nu}\lambda_{(\alpha)\mu}\lambda_{(\beta)\nu} \tag{58}$$

which for the electric field components results in:

$$\begin{aligned} E_x &= E^{(0)}_x - B^{(0)}_y h_{30} + B^{(0)}_x h_{20} \\ E_y &= E^{(0)}_y \left(1 - \frac{1}{2}(h_{00} - h_{22})\right) \\ &- \left(\frac{\alpha x}{2ik} - \frac{\alpha}{2k^2}\right) \mathrm{e}^{ik(x-ct)} - A\mathrm{e}^{ik(x-ct)} + B\mathrm{e}^{-ik(x+ct)} \end{aligned} \tag{59}$$

$$
\begin{aligned}
&+ \quad E_z^{(0)} h_{23} + B_x^{(0)} h_{30} - B_z^{(0)} h_{10} + E_x^{(0)} h_{12} \\
E_z \quad &= \quad E_z^{(0)} \left(1 - \frac{1}{2}(h_{00} - h_{33})\right) \\
&- \quad \left(\frac{\beta x}{2ik} - \frac{\beta}{2k^2}\right) \mathrm{e}^{ik(x-ct)} - C\mathrm{e}^{ik(x-ct)} + D\mathrm{e}^{-ik(x+ct)} \\
&+ \quad E_x^{(0)} h_{13} + B_y^{(0)} h_{10} - B_x^{(0)} h_{20}
\end{aligned}
$$

and for the magnetic field in:

$$
\begin{aligned}
B_x \quad &= \quad B_x^{(0)} + E_z^{(0)} h_{12} - E_y^{(0)} h_{13} \\
B_y \quad &= \quad B_y^{(0)} \left(1 + \frac{1}{2}(h_{33} + h_{11})\right) \\
&+ \quad \frac{\beta x}{2ik} \mathrm{e}^{ik(x-ct)} + C\mathrm{e}^{ik(x-ct)} + D\mathrm{e}^{-ik(x+ct)} \\
B_z \quad &= \quad B_z^{(0)} \left(1 + \frac{1}{2}(h_{22} + h_{11})\right) \\
&- \quad \frac{\alpha x}{2ik} \mathrm{e}^{ik(x-ct)} - A\mathrm{e}^{ik(x-ct)} - B\mathrm{e}^{-ik(x+ct)} - B_y^{(0)} h_{23}
\end{aligned} \tag{60}
$$

### 4.2.2 Wave component solutions

Now that the electromagnetic field components have been put in a non-coordinate basis, the previously mentioned continuity conditions are applied. The values for the arbitrary constants can be computed by requiring that in regions $I$ and $III$ only outgoing waves exist ($A_I = C_I = B_{III} = D_{III} = 0$). This is done in Appendix A. The solutions of the electromagnetic waves in terms of these constants are given below for the three regions under consideration (see Figure 5):

**Region I**

$$\begin{aligned}
E_y &= -B_z = \frac{1}{2}\left(\mathrm{e}^{-2ikL} - 1\right)\mathrm{e}^{-ik(x+ct)} \\
&\times \left\{\frac{\alpha}{2k^2} + \frac{1}{2}(B_z^{(0)} - E_y^{(0)})(\xi_{00} - \xi_{22}) + (E_z^{(0)} + B_y^{(0)})\xi_{23} + (B_x^{(0)}\xi_{30} + E_x^{(0)}\xi_{12})\right\} \\
E_z &= +B_y = \frac{1}{2}\left(\mathrm{e}^{-2ikL} - 1\right)\mathrm{e}^{-ik(x+ct)} \\
&\times \left\{\frac{\beta}{2k^2} - \frac{1}{2}(B_y^{(0)} + E_z^{(0)})(\xi_{00} - \xi_{33}) - (B_x^{(0)}\xi_{20} - E_x^{(0)}\xi_{13})\right\}
\end{aligned} \tag{61}$$

**Region II**

$$\begin{aligned}
E_y &= E_y^{(0)} - \frac{\alpha(x+L)}{2ik}\mathrm{e}^{ik(x-ct)} + \mathrm{e}^{i(\frac{1}{2}\pi - kct)}\sin kx \\
&\times \left\{\frac{\alpha}{2k^2} + \frac{1}{2}(B_z^{(0)} - E_y^{(0)})(\xi_{00} - \xi_{22}) + (E_z^{(0)} + B_y^{(0)})\xi_{23} + (B_x^{(0)}\xi_{30} + E_x^{(0)}\xi_{12})\right\} \\
B_z &= B_z^{(0)} - \frac{\alpha(x+L)}{2ik}\mathrm{e}^{ik(x-ct)} - \mathrm{e}^{i(\frac{1}{2}\pi - kct)}\sin kx \\
&\times \left\{\frac{\alpha}{2k^2} + \frac{1}{2}(B_z^{(0)} - E_y^{(0)})(\xi_{00} - \xi_{22}) + (E_z^{(0)} + B_y^{(0)})\xi_{23} + (B_x^{(0)}\xi_{30} + E_x^{(0)}\xi_{12})\right\} \\
E_z &= E_z^{(0)} - \frac{\beta(x+L)}{2ik}\mathrm{e}^{ik(x-ct)} + \mathrm{e}^{i(\frac{1}{2}\pi - kct)}\sin kx \\
&\times \left\{\frac{\beta}{2k^2} - \frac{1}{2}(E_z^{(0)} + B_y^{(0)})(\xi_{00} - \xi_{33}) - (B_x^{(0)}\xi_{20} - E_x^{(0)}\xi_{13})\right\} \\
B_y &= B_y^{(0)} + \frac{\beta(x+L)}{2ik}\mathrm{e}^{ik(x-ct)} + \mathrm{e}^{i(\frac{1}{2}\pi - kct)}\sin kx \\
&\times \left\{\frac{\beta}{2k^2} - \frac{1}{2}(E_z^{(0)} + B_y^{(0)})(\xi_{00} - \xi_{33}) - (B_x^{(0)}\xi_{20} - E_x^{(0)}\xi_{13})\right\}
\end{aligned} \tag{62}$$

**Region III**

$$\begin{aligned}
E_y &= B_z = -\frac{\alpha L}{2ik}\mathrm{e}^{ik(x-ct)} \\
E_z &= -B_y = -\frac{\beta L}{2ik}\mathrm{e}^{ik(x-ct)}
\end{aligned} \tag{63}$$

If the second part of the device is set up the same way as before, such that the length of this part is a half integer multiple of the wavelength ($L = n\pi/k = \frac{1}{2}n\lambda$), the retreating electromagnetic waves vanish, and what remains are just two linearly polarized electromagnetic waves propagating in the positive $x$-direction. This is just what one expects, as it is the reverse process of the EMW to GW conversion in the first part of the detector (in accordance to the remark above (33)).

### 4.3 Efficiency

It is possible to set up an arrangement where one generates an electromagnetic wave, which produces a gravitational wave. The remaining light is then screened off, while the gravitational wave is reconverted into an electromagnetic wave. A measurement of the thus produced electromagnetic intensity would proof the intermediate existence of a gravitational wave. This intensity, in the magnetostatic case, is given by:

$$\begin{aligned} W &= \frac{A^2}{4\pi}\left(\frac{GLL'}{c^4}\right)^2\left[(B_z B_z^{(0)} + B_y B_y^{(0)})^2 + (B_y B_z^{(0)} - B_z B_y^{(0)})^2\right] \\ &= \frac{A^2}{4\pi}\left(\frac{G}{c^4}\right)^2 (LB_z)^4 \end{aligned} \tag{64}$$

where in the last line the length and the electromagnetic field of the two regions are taken to be equal. One can see that the final intensity depends on the fourth power of both the length and the magnetic field used (and on the square of the intensity of the initial electromagnetic field $A$). Still, the smallness of the factor $(G/c^4)^2$ does not make this a very feasible terrestial detection device at present.

However, it will be shown in the next few sections, that the astrophysical relevance of the discussed GW to EMW conversion is significant. In most cases, GW to EMW conversions will occur (and not the reverse process) the efficiency of which is given by the square root of (64). For a magnetostatic background field, this, of course, results in the same efficiency as in Sec. 3 and Sec. 4.1.1.

The main purpose of the calculations in this section was to rigorously derive the characteristics of GW $\Leftrightarrow$ EMW conversions in the most general circumstances.

# 5 Magnetars and binary mergers

In the previous sections, the possibility of converting gravitational wave energy into light in the presence of a static, homogeneous electromagnetic field was discussed. Such a conversion is interesting for two reasons. First, it could be relevant for the energy excess in the electromagnetic spectrum of very energetic phenomena such as gamma ray bursts. The reason for this is that a sub-class of GRBs is supposed to be powered by binary mergers, which release most of their energy in GW and not in EM radiation.

Secondly, it could be a useful indirect detection device of gravitational waves, because EMWs are much easier to detect then GWs.

One should therefore look for astrophysical sources with the following ingredients:

1. Obviously, one would like to have a source that produces gravitational waves that carry large amounts of energy, so that even if only a small fraction is transferred to EMWs, the result is significant,

2. The source has to produce GWs with relatively high frequencies ($\approx 10$ kHz). Otherwise, the EMWs have a frequency (the same as the GW) below the interstellar plasma frequency and will be absorbed before reaching earth,

3. The interaction will only take place in either an extremely strong magnetic field, or a (weaker) field that prevails over extremely large distances (i.e. a primordial field).

4. The region of interaction is a vacuum or a thin plasma, so that one can neglect the difference of the dispersion relation for the EMW from vacuum and GW-dissipations.

The most promising candidate, satisfying these demands, is the so called magnetar, a compact neutron star formed by an a-symmetic supernova collapse, a merger or collision of two neutron stars as studied by Kokkotas et. al. ([31]-[20]).

What remains after such a collapse, is a violently oscillating neutron star, quaking in several distinct quasi-normal modes. Post-Newtonian fluid modes, such as the $f$-mode (fundamental mode), $p$-mode (pressure as restoring force) and $g$-modes (gravity as restoring force), are dominated by the fundamental mode with typical frequencies of $1-2$Hz. The socalled $w$ave modes, on the other hand, are purely general relativistic modes. These are not fluid modes, but GWs due to strong space-time dynamics, with typical frequencies of $8-12$kHz (and up to 38kHz). This means that the $w$ modes might be able to produce EMWs in the right frequency domain (even without photon acceleration).

The energy radiated by these GWs is a fraction of a solar rest mass energy and is radiated in a small damping time in axisymmetric waves, producing a very large gravitational flux density.

Finally, the typical magnetic fields of such magnetars are the largest ever encountered: $\approx 10^{16}$ Gauss. If one now assumes that the far field of the magnetar can be regarded as a vacuum (Sec. 5.1) or a thin plasma (Sec. 6), all our conditions are satisfied.

The next subsection gives an estimate of the produced electromagnetic energy flux due to gravitational waves in the presence of a magnetar using the results of Sec. 3 and Sec. 4.2. Then in Sec. 6 the same will be done for a thin plasma using a slightly different approach in which the Maxwell equations are already put in a non-coordinate basis before solving them. This allows easy comparison of the current densities in the plasma with the gravitational effects induced by the GW.

### 5.1 Estimate of efficiency for magnetar

In Figure 6, the (far) magnetic field of a magnetar with mass $M = M_\odot$, radius $R_1 \approx R_S = 2GM/c^2 = 3$ km and surface field $B_1 = 10^{16}$Gauss is shown.

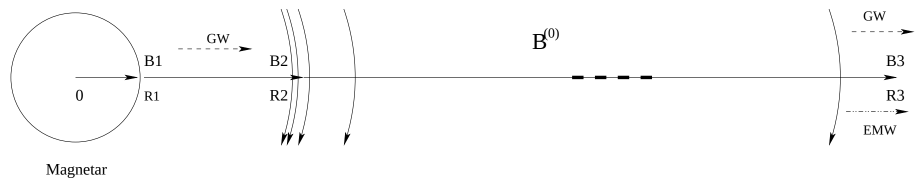


Figure 6: Gravitons to photons around magnetar.

The magnetic field of such a neutron star will be that of a dipole out to the light cylinder and a spherical $1/r$ decay beyond that. Since, however, the studies [31]-[20] involve a non-rotating star, only the dipole field will be of importance.

Furthermore, one can only speak of a GW in the radiation zone, for $r \gg R_1$. In this estimate, following Brodin, Marklund et. al. [11], it will be assumed that the interaction becomes effective around $R_2 = 60R_S = 180$km.

$$B(r) \quad = \quad B_1 \left(\frac{R_1}{r}\right)^3 \qquad , \qquad R_1 < r < R_3 \tag{65}$$

To compute the amplitudes of the generated EMWs, use (32) or (46). To apply this formula to the reverse case, realize that the efficiency of GW to EMW conversion in a static magnetic field is the same as for the reverse case, since the relation connecting $\phi_{\mu\nu} \sim h_{\mu\nu}$ and $F_{\mu\nu}$ is linear and symmetric under time reversal. So, for $c = G = 1$, $b = \bar{E} = -\bar{B}$, and $a(z)/k = \bar{h}$, where $\bar{h}$ is the dimensionless effective amplitude of a +-polarised plane GW, the efficiency is:

$$\sqrt{\alpha'} \quad = \quad \mid \frac{\bar{E}}{\bar{h}} \mid = \frac{-ik}{2} \int \hat{B}(z) dz \tag{66}$$

And using the GW amplitude as calculated by Kokkotas et. al. in [1]:

$$\bar{h} \quad = \quad 10^{-21} \left(\frac{E}{10^{-6} M_\odot c^2}\right)^{1/2} \left(\frac{10\text{kHz}}{f}\right)^{1/2} \left(\frac{50\text{kpc}}{r}\right) \tag{67}$$

the EMW amplitudes can be calculated:

$$\begin{aligned} E_y(r,t) \quad &= \quad -B_x(r,t) \qquad (68) \\ &= \quad -\frac{ik\bar{h}B_1}{2} \left( \int_{R_2}^{r} \left(\frac{R_1}{r'}\right)^3 dr' \right) \mathrm{e}^{ik(r-t)} \\ &= \quad ik\bar{h}B_1R_1^3 \left(\frac{1}{r^2} - \frac{1}{R_2^2}\right) \mathrm{e}^{ik(r-t)} \end{aligned}$$

The most energetic magnetars, with $E = 10^{-2}M_{\odot}c^2$ will have incident GWs at $r = 180$km with an effective amplitude of $\bar{h} = 0.001$. Putting this into the equation for the produced electromagnetic waves and choosing some cut-off distance $R_3$ where the magnetic field, or just the coupling, becomes negligible (or $1/R_3^2$ is just negligible with respect to $1/R_2^2$), the result is:

$$E_{y\max} = -B_{x\max} \sim 10^{12}\mathrm{V/m} \tag{69}$$

using $k = \omega$ with $\omega/2\pi = 10$kHz.

This large value results mainly because of the extremely large magnetic field, which is four orders of magnitude larger than for instance the field of a neutron star binary.

## 5.2 Comparison with neutron star binary

If, in (66), a close binary just before merger is considered as a source, with the same distance scales, but a magnetic surface field of $10^{12}$Gauss, an effective (Newtonian) GW amplitude of $4 \cdot 10^{-4}$ and a GW frequency of $2 \cdot 10^3$rad/s (caused by the rotation of the binary instead of the magnetar star-quakes), the EMW amplitude is still very significant with 50MV/m.

The frequency of the latter EMW, however, causes these waves to be damped by the interstellar plasma, whereas the former could easily be detected by a space based detector (avoiding the ionospheric cutoff), as long as the source resides close to our galaxy. In fact, for the extreme case discussed here (maximum magnetic field and maximum gravitational wave energy), the EMW amplitude reaching earth from a source at a distance of $\approx 50$kpc would be as much as 1V/m if the EMW only decayed by spherical attenuation.

It has to be realized, though, that the values mentioned here are probably highly exaggerated when one takes into account all the real dynamics around a magnetar or neutronstar binary. More research has to be done to deal with spherically decaying GWs, non-uniform plasmas etc.

## 5.3 Improvements to calculation

As a first step towards a more realistic astrophysical setting, the vacuum has to be replaced by a thin plasma. In the next sections (6.1–6.4), it becomes apparent, that the interaction is hardly effected by the presence of such a plasma. The presence of a plasma induces a slight wavenumber offset as compared to the vacuum waves, but as long as the coherence lengths of the GW and EMW are longer than the extension of the static magnetic field, the equations for $E_y(r,t) = -B_x(r,t)$ will remain the same as for the vacuum case. The main importance of adding a plasma is to either damp or excite the EMWs to higher frequencies.

In Sec. 7, the plasma will be considered in the magnetohydrodynamic approximation of a perfectly conducting ideal fluid with pressure gradients. A dispersion relation is derived in Sec. 7.4, in search of plasma waves such as Alfvén waves, magnetosonic waves and slow and fast modes, excited by the GWs.

# 6 Plasma calculations

To obtain the expressions for the EMW produced by a GW traveling through a plasma in more or less the same fashion as in Section 4, the Maxwell equations are written in terms of a general local orthonormal frame. This will be done in the next section 6.1. In Sec. 6.2 the ONF will be specified for a plane polarized GW.

## 6.1 Non-coordinate Maxwell equations

The general, coordinate, decomposition of the Faraday tensor with respect to an observer moving with a velocity $u^a$, satisfying $u_a u^a = -1$ is:[5]

$$\begin{aligned} F^{ab} &= u^a E^b - u^b E^a + \epsilon^{abc} B_c \quad \text{or} \\ E_a &= F_{0a} \quad , \quad B_a = \frac{1}{2}\epsilon_{abc} F^{bc} \end{aligned} \tag{70}$$

where $\epsilon_{abc} = u^d \eta_{abcd}$ is the volume element on hypersurfaces orthogonal to $u^a$, which is skew symmetric in all its indices and satisfies $\epsilon_{123} = \eta_{0123} = \sqrt{|g|} = 1$.

To evaluate the Maxwell equations in a locally inertial non-coordinate system, again an orthonormal tedrad is used: $\mathbf{e}_0 = u$ and $\mathbf{e}_\alpha = \nabla_\alpha$ (with $e_a{}^i e_b{}^j g_{ij} = \eta_{ab}$). In this ONF the covariant Faraday tensor is just:

$$F_{ab} = u_a E_b - u_b E_a + \epsilon_{abc} B^c \tag{71}$$

and the covariant derivatives are defined, completely analogous to the usual tensor form in terms of Christoffel symbols, as:

$$\begin{aligned} \nabla_a F_{bc} &= \mathbf{e}_a(F_{bc}) - \Gamma^d{}_{ba} F_{dc} - \Gamma^d{}_{ca} F_{bd} \\ \nabla_a F^{bc} &= \mathbf{e}_a(F^{bc}) + \Gamma^b{}_{da} F^{dc} + \Gamma^c{}_{da} F^{bd} \end{aligned} \tag{72}$$

where in the non-coordinate desciption, the Christoffel symbols are replaced by the socalled *Ricci rotation coefficients*. These are defined by:

$$\Gamma^c{}_{ab} = e^c{}_i e_b{}^j \nabla_j e_a{}^i \quad \Leftrightarrow \quad \nabla_{\mathbf{e}_b} \mathbf{e}_a = \Gamma^c{}_{ab} \mathbf{e}_c \quad \text{and} \quad \Gamma_{(ab)c} = 0 \tag{73}$$

The general form of the inhomogeneous Maxwell equations in the ONF can now be expressed as:

$$\begin{aligned} \nabla_a F^{ba} &= \mathbf{e}_a(F^{ba}) + \Gamma^b{}_{da} F^{da} + \Gamma^a{}_{da} F^{bd} \\ &= \mathbf{e}_a(u^b E^a) - \mathbf{e}_a(u^a E^b) + \epsilon^{bac} \mathbf{e}_a(B_c) \end{aligned} \tag{74}$$

[5] From now $c = 1$, roman indices are $0, 1, 2, 3$ and greek indices $1, 2, 3$ following the conventions in [11].

$$
\begin{aligned}
&+ \quad \Gamma^{b}{}_{da} u^{d} E^{a} - \Gamma^{b}{}_{da} u^{a} E^{d} + \epsilon^{dac} \Gamma^{b}{}_{da} B_{c} \\
&+ \quad \Gamma^{a}{}_{da} u^{b} E^{d} - \Gamma^{a}{}_{da} u^{d} E^{b} + \epsilon^{bdg} \Gamma^{a}{}_{da} B_{g} \\
&= \quad 4\pi (j_m)^{b}
\end{aligned}
$$

and similarly for the homogeneous equations. In evaluating these expressions, one has to realize that:

1. $E^a = E^\alpha$ and $B^a = B^\alpha$, so all terms in $E^0$ or $B^0$ vanish,
2. the time axis is chosen such that $\mathbf{e}_0 = u$, so the observer is at rest with respect to the ONF and only $u^0 = 1$ is nonvanishing (all terms $u^\alpha$ are zero),
3. $\epsilon^{0ab} = u_d \eta^{0abd} = u_0 \eta^{0ab0} = 0$ and similarly $\epsilon^{a0b} = \epsilon^{ab0} = 0$,
4. since the rotation coefficients are skew symmetric in the first two indices $\Gamma_{aab} = 0$.

**Inhomogeneous equations** Using these 'selection rules', one finds for the first inhomogeneous Maxwell equation ($\nabla_a F^{0a} = 4\pi\rho$):

$$
\begin{aligned}
\nabla_a F^{0a} &= \mathbf{e}_a(E^a) - \Gamma^{0}{}_{\delta 0} E^{\delta} + \Gamma^{0}{}_{\delta 0} E^{\delta} + \Gamma^{\alpha}{}_{\delta\alpha} E^{\delta} + \epsilon^{\delta\alpha\gamma} \Gamma^{0}{}_{\delta\alpha} B_{\gamma} \\
4\pi\rho_m & \\
\nabla \cdot \mathbf{E} &= (-\Gamma^{\alpha}{}_{\beta\alpha} E^{\beta} - \epsilon^{\alpha\beta\gamma} \Gamma^{0}{}_{\alpha\beta} B_{\gamma}) + 4\pi\rho_m \\
&= \rho_E + 4\pi\rho_m
\end{aligned} \tag{75}
$$

where the last equation *defines* the extra 'charge density' $\rho_E$. The other three inhomogeneous equations follow from $\nabla_a F^{\beta a} = 4\pi j^{\beta}$:

$$
\begin{aligned}
\nabla_a F^{\beta a} &= -\mathbf{e}_0(E^{\beta}) + \epsilon^{\beta\alpha\gamma} \mathbf{e}_{\alpha}(B_{\gamma}) \\
&+ (\Gamma^{\beta}{}_{0\alpha} E^{\alpha} - \Gamma^{\beta}{}_{\delta 0} E^{\delta}) - \Gamma^{\alpha}{}_{0\alpha} E^{\beta} \\
&+ (\epsilon^{\delta\alpha\gamma} \Gamma^{\beta}{}_{\delta\alpha} + \epsilon^{\beta\delta\gamma} \Gamma^{\alpha}{}_{\delta\alpha}) B_{\gamma} + \epsilon^{\beta\delta\gamma} \Gamma^{0}{}_{\delta 0} B_{\gamma} \\
\mathbf{e}_0(\mathbf{E}) - \nabla \times \mathbf{B} &= (\Gamma^{\alpha}{}_{0\beta} - \Gamma^{\alpha}{}_{\beta 0}) E^{\beta} - \Gamma^{\beta}{}_{0\beta} E^{\alpha} \\
&+ \epsilon^{\alpha\beta\gamma} (\Gamma^{0}{}_{\beta 0} B_{\gamma} + \Gamma^{\delta}{}_{\beta\gamma} B_{\delta}) - 4\pi \mathbf{j}_m \\
&= -\mathbf{j}_{\mathbf{E}} - 4\pi \mathbf{j}_m
\end{aligned} \tag{76}
$$

where the intermediate result that $(\epsilon^{\delta\alpha\gamma}\Gamma^{\beta}{}_{\delta\alpha} + \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\alpha})B_{\gamma} = \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\gamma}B_{\alpha}$ is proved in Appendix B, and where the last equation defines $\mathbf{j}_E$.

**Homogeneous equations** The first homogeneous Maxwell equation follows from chosing $(abc) = (\alpha\beta\gamma)$:

$$
\begin{aligned}
\nabla_{(\alpha} F_{\beta\gamma)} &= \nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} \\
&= \epsilon_{\beta\gamma\delta}\mathbf{e}_\alpha(B^\delta) + \epsilon_{\alpha\beta\delta}\mathbf{e}_\gamma(B^\delta) + \epsilon_{\gamma\alpha\delta}\mathbf{e}_\beta(B^\delta) \\
&- \Gamma^d{}_{\beta\alpha}F_{d\gamma} - \Gamma^d{}_{\alpha\gamma}F_{d\beta} - \Gamma^d{}_{\gamma\beta}F_{d\alpha} \\
&- \Gamma^d{}_{\gamma\alpha}F_{\beta d} - \Gamma^d{}_{\beta\gamma}F_{\alpha d} - \Gamma^d{}_{\alpha\beta}F_{\gamma d} \\
-\mathbf{e}_\delta(B^\delta) &= \Gamma^0{}_{\alpha\beta}E_\gamma + \Gamma^0{}_{\gamma\alpha}E_\beta + \Gamma^0{}_{\beta\gamma}E_\alpha \\
&- \Gamma^0{}_{\beta\alpha}E_\gamma - \Gamma^0{}_{\alpha\gamma}E_\beta - \Gamma^0{}_{\gamma\beta}E_\alpha \\
&+ \epsilon_{\gamma\delta\nu}\Gamma^\delta{}_{\beta\alpha}B^\nu + \epsilon_{\beta\delta\nu}\Gamma^\delta{}_{\alpha\gamma}B^\nu + \epsilon_{\alpha\delta\nu}\Gamma^\delta{}_{\gamma\beta}B^\nu \\
&- \epsilon_{\alpha\delta\nu}\Gamma^\delta{}_{\beta\gamma}B^\nu - \epsilon_{\gamma\delta\nu}\Gamma^\delta{}_{\alpha\beta}B^\nu - \epsilon_{\beta\delta\nu}\Gamma^\delta{}_{\gamma\alpha}B^\nu \\
&= \epsilon^{\alpha\beta\gamma}\Gamma^0{}_{\alpha\beta}E_\gamma - \epsilon^{\alpha\beta\gamma}\epsilon_{\gamma\delta\nu}\Gamma^\delta{}_{\alpha\beta}B^\nu \\
&= \epsilon^{\alpha\beta\gamma}\Gamma^0{}_{\alpha\beta}E_\gamma + \Gamma^\beta{}_{\nu\beta}B^\nu \qquad (+\Gamma^\alpha{}_{\alpha\beta}B^\beta = 0) \\
\nabla\cdot\mathbf{B} &= -\Gamma^\alpha{}_{\beta\alpha}B^\beta - \epsilon^{\alpha\beta\gamma}\Gamma^0{}_{\alpha\beta}E_\gamma = \rho_B
\end{aligned}
\tag{77}
$$

The other homogeneous Maxwell equations follow from chosing $(abc) = (0\alpha\beta)$:

$$
\begin{aligned}
\nabla_{(0} F_{\alpha\beta)} &= \nabla_0 F_{\alpha\beta} + \nabla_\beta F_{0\alpha} + \nabla_\alpha F_{\beta 0} \\
&= (\mathbf{e}_\beta(E_\alpha) - \mathbf{e}_\alpha(E_\beta)) + \epsilon_{\alpha\beta\gamma}\mathbf{e}_0(B^\gamma) \\
&- (\Gamma^d{}_{\alpha 0}F_{d\beta} + \Gamma^d{}_{\beta 0}F_{\alpha d}) \\
&- (\Gamma^d{}_{0\beta}F_{d\alpha} + \Gamma^d{}_{\alpha\beta}F_{0d}) - (\Gamma^d{}_{\beta\alpha}F_{d0} + \Gamma^d{}_{0\alpha}F_{\beta d}) \\
-\mathbf{e}_0(\mathbf{B}) + \nabla\times\mathbf{E} &= -(\Gamma^0{}_{\alpha 0}E_\beta - \Gamma^0{}_{\beta 0}E_\alpha) - \Gamma^0{}_{0\beta}E_\alpha \\
&- (\Gamma^\delta{}_{\alpha\beta}E_\delta - \Gamma^\delta{}_{\beta\alpha}E_\delta) - \Gamma^0{}_{0\alpha}E_\beta \\
&+ (\epsilon_{\beta\delta\gamma}\Gamma^\delta{}_{\alpha 0} - \epsilon_{\alpha\delta\gamma}\Gamma^\delta{}_{\beta 0})B^\gamma \\
&+ (\epsilon_{\alpha\delta\gamma}\Gamma^\delta{}_{0\beta} - \epsilon_{\beta\delta\gamma}\Gamma^\delta{}_{0\alpha})B^\gamma \\
&= -\epsilon^{\alpha\beta\gamma}(\Gamma^0{}_{\beta 0}E_\gamma + \Gamma^\delta{}_{\beta\gamma}E_\delta) \\
&+ (\epsilon^{\nu\beta\alpha}\epsilon_{\alpha\delta\gamma}\Gamma^\delta{}_{0\beta} - \epsilon^{\nu\beta\alpha}\epsilon_{\alpha\delta\gamma}\Gamma^\delta{}_{0\beta})B^\gamma \\
\mathbf{e}_0(\mathbf{B}) - \nabla\times\mathbf{E} &= \epsilon^{\alpha\beta\gamma}(\Gamma^0{}_{\beta 0}E_\gamma + \Gamma^\delta{}_{\beta\gamma}E_\delta) \\
&+ (\Gamma^\alpha{}_{0\beta} - \Gamma^\alpha{}_{\beta 0})B^\beta - \Gamma^\beta{}_{0\beta}B^\alpha \\
&= -\mathbf{j}_B
\end{aligned}
\tag{78}
$$

## 6.2 Gravity induced charge and current densities

In this section, the ONF used in the non-coordinate Maxwell equations (75)-(78), is specified to a plane, +-polarized GW with metric: $\mathrm{diag}(-1,(1+h),(1-h),1)$ (compare with (12)). The natural tedrad for this system is, as one can easily verify by applying the orthonormality condition (56):

$$\begin{aligned}
e_{(0)}{}^{i} &= (1,0,0,0)) \\
e_{(1)}{}^{i} &= (0,(1-\frac{1}{2}h),0,0) \\
e_{(2)}{}^{i} &= (0,0,(1+\frac{1}{2}h),0) \\
e_{(3)}{}^{i} &= (0,0,0,1)
\end{aligned} \tag{79}$$

In this ONF the only nonzero rotation coeffiecients are $\Gamma^{1}{}_{13} = -\Gamma^{2}{}_{23} = \frac{1}{2}\frac{\partial h}{\partial z}$ and $\Gamma^{1}{}_{10} = -\Gamma^{2}{}_{20} = -\frac{1}{2}\frac{\partial h}{\partial t}$, since only the $z$ and $t$ derivatives of the $\mathbf{e}_2$ and $\mathbf{e}_3$ vectors contribute.

This imediately tells one that the gravity induced charge densities vanish. The only components that remain for the induced current densities are:

$$\begin{aligned}
\mathbf{j}_E &= \left[\Gamma^{\alpha}{}_{\beta 0}E^{\beta} - \epsilon^{\alpha\beta\gamma}\Gamma^{\delta}{}_{\beta\gamma}B_{\delta}\right]\mathbf{e}_{\alpha} \\
\mathbf{j}_B &= \left[\Gamma^{\alpha}{}_{\beta 0}B^{\beta} - coo\epsilon^{\alpha\beta\gamma}\Gamma^{\delta}{}_{\beta\gamma}E_{\delta}\right]\mathbf{e}_{\alpha}
\end{aligned} \tag{80}$$

Explicitely, the second term in the equation for $\mathbf{j}_E$ for $\alpha = 1$, $\beta = 2$ and $\gamma = 3$ becomes[6]:

$$\begin{aligned}
(\Gamma^{\delta}{}_{23} - \Gamma^{\delta}{}_{32})B_{\delta} &= \Gamma^{\delta}{}_{23}B_{\delta} = (0,(1+\frac{1}{2}h),0)\frac{\partial}{\partial z}\begin{pmatrix}(1+\frac{1}{2}h)B_x \\ (1-\frac{1}{2}h)B_y \\ B_z\end{pmatrix} \\
&= -\frac{1}{2}\dot{h}B_y
\end{aligned} \tag{81}$$

whereas for $\alpha = 2$, $\beta = 3$ and $\gamma = 1$:

$$\begin{aligned}
(\Gamma^{\delta}{}_{31} - \Gamma^{\delta}{}_{13})B_{\delta} &= -\Gamma^{\delta}{}_{13}B_{\delta} = -((1-\frac{1}{2}h),0,0)\frac{\partial}{\partial z}\begin{pmatrix}(1+\frac{1}{2}h)B_x \\ (1-\frac{1}{2}h)B_y \\ B_z\end{pmatrix} \\
&= -\frac{1}{2}\dot{h}B_x
\end{aligned} \tag{82}$$

Similarly from the equation for $\mathbf{j}_B$ (replacing $\mathbf{B} \to -\mathbf{E}$) it follows that:

$$\begin{aligned}
-\Gamma^{\delta}{}_{23}E_{\delta} &= \frac{1}{2}\dot{h}E_y \quad \text{and} \\
-\Gamma^{\delta}{}_{31}E_{\delta} &= \frac{1}{2}\dot{h}E_x
\end{aligned} \tag{83}$$

[6] using $\dot{h} = \frac{\partial h}{\partial z} = -\frac{\partial h}{\partial t}$.

and for the other components of $\mathbf{j}_E$:

$$\begin{aligned} \Gamma^1{}_{\beta 0} E^\beta &= -\frac{1}{2}\frac{\partial h}{\partial t} E_x \\ \Gamma^2{}_{\beta 0} E^\beta &= \frac{1}{2}\frac{\partial h}{\partial t} E_y \end{aligned} \tag{84}$$

and for $\mathbf{j}_B$:

$$\begin{aligned} \Gamma^1{}_{\beta 0} B^\beta &= -\frac{1}{2}\frac{\partial h}{\partial t} B_x \\ \Gamma^2{}_{\beta 0} B^\beta &= \frac{1}{2}\frac{\partial h}{\partial t} B_y \end{aligned} \tag{85}$$

Putting things together:

$$\begin{aligned} \mathbf{j}_E &= -\frac{1}{2}\dot{h}\begin{pmatrix} B_y - E_x \\ B_x + E_y \\ 0 \end{pmatrix} \\ \mathbf{j}_B &= \frac{1}{2}\dot{h}\begin{pmatrix} B_x + E_y \\ B_y - E_x \\ 0 \end{pmatrix} \end{aligned} \tag{86}$$

The Maxwell equations in the presence of a gravitational wave are summarized below in their final form for future reference.

$$\boxed{\begin{array}{rcll} \nabla\cdot\mathbf{E} &=& 4\pi\rho_m & (\text{POISSON}) \\ \frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B} &=& -4\pi\mathbf{j}_m - \mathbf{j}_E & (\text{AMPERE}) \\ \nabla\cdot\mathbf{B} &=& 0 & (\text{NO MONOPOLES}) \\ \frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} &=& -\mathbf{j}_B & (\text{FARADAY}) \end{array}}$$

With $\mathbf{j}_E$ and $\mathbf{j}_B$ defined in (86).

### 6.3 Energy-momentum conservation

To evaluate the Maxwell equations, or to find electromagnetic wave equations, one still needs an expression for the matter current. This expression follows from the linearized equation of motion. In this case, for a one component plasma where the ion velocity is negligible compared to the electron velocity, one can use the energy momentum conservation equation (see [16] or [23]).

The matter part of the stress energy tensor, for a pressure free 'dust-like' plasma, is (see [36] and [28]) $T_{\rm M}^{ab} = \rho V^a V^b$ (where $V^a = (1, \mathbf{v})$ is the non-relativistic fluid velocity for $\gamma \ll 1$). The electromagnetic part is not needed here because of the useful identity $\nabla_b T_{\rm EM}^{ab} = -F^{ab}(j_{\rm M})_b$. Thus the conservation equations follow from:

$$\nabla_b(mnV^aV^b) - qn(u^aE^b - u^bE^a + \epsilon^{abc}B_c)V_b = 0 \tag{87}$$

or, in a $1+3$ split,

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) &= -n(\Gamma^\alpha{}_{0\alpha} + \Gamma^\alpha{}_{00}v_\alpha + \Gamma^\alpha{}_{\beta\alpha}v^\beta) \qquad (88)\\ \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} &= \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - (\Gamma^\alpha{}_{00} + (\Gamma^\alpha{}_{0\beta} + \Gamma^\alpha{}_{\beta 0})v^\beta + \Gamma^\alpha{}_{\beta\gamma}v^\beta v^\gamma) \end{aligned}$$

In the specified metric, these equations can be linearized around the unperturbed plasma state: $\mathbf{E} = 0$, $\mathbf{v}^0 = 0$ and $\bar{\mathbf{v}}(z,t) = \bar{\mathbf{v}}(z)\mathrm{e}^{-i\omega t}$, where all the barred quantities are first order amplitudes. In that case, all the terms containing the rotation coefficients are either zero or quadratic in $\bar{\mathbf{h}}$, which reduces the equation of motion (88) to:

$$\frac{\partial\bar{\mathbf{v}}}{\partial t} = -i\omega\bar{\mathbf{v}} = \frac{q}{m}(\bar{\mathbf{E}} + \bar{\mathbf{v}}\times\hat{\mathbf{B}}) \tag{89}$$

Now take $\mathbf{j}_m = qn_0\mathbf{v}$ (with $n_0$ the unperturbed electron number density) and introduce the following relevant frequencies:

$$\begin{aligned} \text{CYCLOTRON FREQUENCY:} \quad &\Rightarrow \quad \omega_c = \frac{q\hat{B}}{m} \qquad (90)\\ \text{PLASMA FREQUENCY:} \quad &\Rightarrow \quad \omega_p = \sqrt{\frac{4\pi n_0 q^2}{m}}\\ \text{UPPER HYBRID FREQUENCY:} \quad &\Rightarrow \quad \omega_h = \sqrt{\omega_c^2 + \omega_p^2} \end{aligned}$$

and solve (89) for $\bar{\mathbf{v}}$ or $\bar{\mathbf{j}}_{\rm M}$:

$$\begin{aligned} (\bar{j}_m)_x &= \frac{1}{4\pi}\frac{\omega_p^2}{\omega}i\bar{E}_x \qquad (91)\\ (\bar{j}_m)_y &= \frac{1}{4\pi}\frac{\omega_p^2}{\omega}\left(i\bar{E}_y + \frac{\omega_c}{\omega}\bar{E}_z\right)\left(\frac{\omega^2}{\omega^2 - \omega_c^2}\right)\\ (\bar{j}_m)_z &= \frac{1}{4\pi}\frac{\omega_p^2}{\omega}\left(i\bar{E}_z - \frac{\omega_c}{\omega}\bar{E}_y\right)\left(\frac{\omega^2}{\omega^2 - \omega_c^2}\right) \end{aligned}$$

The component needed in the next section is:

$$4\pi i\omega(\bar{j}_{\mathrm{M}})_y = -\omega_p^2\left(\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2}\right)\bar{E}_y \tag{92}$$

## 6.4 Exact calculation for the plasma waves

Now that the Maxwell equations are put in an elegant and transparent form, it is straightforward to obtain wave equations for an EMW excited by a GW passing through a static magnetic background field $\mathbf{B}^{(0)} = (\hat{B}(z), 0, 0)$. For the electric wave components, take the curl of Faraday's law (78) and substitute it into Ampère's law, differentiated with respect to time. And to find magnetic components, proceed the other way around. Thus, one finds:

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E} &= -\frac{\partial}{\partial t}(4\pi\mathbf{j}_m+\mathbf{j}_E)-\nabla\times\mathbf{j}_B \\ \left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{B} &= \nabla\times(4\pi\mathbf{j}_m+\mathbf{j}_E)-\frac{\partial\mathbf{j}_B}{\partial t}\end{aligned} \tag{93}$$

To solve these equations assume the GW to be monochromatic and let the frequency of all perturbations coincide with that of the driver, leaving the spatial dependence free for the moment, i.e.:

$$\begin{aligned}\mathbf{h} &= \bar{\mathbf{h}}\mathrm{e}^{ik(z-t)} \\ \mathbf{E} &= \bar{\mathbf{E}}(z)\mathrm{e}^{-i\omega t} \\ \mathbf{j}_m &= \bar{\mathbf{j}}_m(z)\mathrm{e}^{-i\omega t} \\ \mathbf{B} &= \hat{\mathbf{B}}^{(0)}(z)\mathbf{e}_1+(\bar{\mathbf{B}})^{(1)}(z)\mathrm{e}^{-i\omega t}\end{aligned} \tag{94}$$

Dropping all second order terms in the perturbation (which are in effect just the current density terms in $\bar{\mathbf{h}}\cdot\bar{\mathbf{E}}$), the relevant wave components are:

$$\left(\frac{\partial^2}{\partial z^2}+\omega^2\right)\bar{E}_y+4\pi i\omega(\bar{j}_m)_y = -\left(k^2\hat{B}-\frac{1}{2}ik\frac{\partial\hat{B}}{\partial z}\right)\bar{h}\mathrm{e}^{ikz}\equiv F(z) \tag{95}$$

$$\left(\frac{\partial^2}{\partial z^2}+\omega^2\right)\bar{B}_x+4\pi\frac{\partial}{\partial z}(\bar{j}_m)_y = \left(k^2\hat{B}-\frac{1}{2}ik\frac{\partial\hat{B}}{\partial z}\right)\bar{h}\mathrm{e}^{ikz} = -F(z) \tag{96}$$

where $F(z)$ is a function of $z$ through the specific form of the static background field.

The expression for $\hat{E}_y$ is now easy to evaluate using (92), while $\hat{B}_x$ is harder to find, since the $z$-dependence of $(j_m)_y$ has not been specified. $\hat{B}_x$ can, however, be deduced from $\hat{E}_y$, as will be done below.

**Solution for $\hat{E}_y$:** Inserting (92) in (95), leads to an inhomogeneous differential equation for **E**:

$$\left(\frac{\partial^2}{\partial z^2} + \omega^2 - \omega_p^2 \left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}\right)\right) \bar{E}_y = F(z) \tag{97}$$

To find the solution of this equation, one first determines the solution of the homogeneous equation, taking $F(z) = 0$, and its *Wronskian*:

$$\begin{aligned} \bar{E}_y^h(z) &= A\mathrm{e}^{ik_l z} + B\mathrm{e}^{-ik_l z} \\ W &= \det \begin{vmatrix} A\mathrm{e}^{ik_l z} & B\mathrm{e}^{-ik_l z} \\ ik_l A\mathrm{e}^{ik_l z} & -ik_l B\mathrm{e}^{-ik_l z} \end{vmatrix} = -2ik_l AB \end{aligned} \tag{98}$$

In (98), $k_l$ is the wavenumber of the EMW satisfying the dispersion equation $k_l^2 = \omega^2 - (\Delta\omega)^2$ which is slightly shifted compared to the dispersion relation in vacuum. This shift is given by:

$$\Delta\omega = \sqrt{\omega_p^2 \left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}\right)} \ll \omega \tag{99}$$

The general solution of the inhomogeneous equation is then given by:

$$\bar{E}_y(z) = \frac{i}{2k_l}\left(\mathrm{e}^{ik_l z}\int_0^z \mathrm{e}^{-ik_l z'} F(z')dz' + \mathrm{e}^{-ik_l z}\int_z^L \mathrm{e}^{ik_l z'} F(z')dz'\right) \tag{100}$$

Here, as in Sec. 3, the assumptions have been made that the static background field is restricted to a region $0 < z < L$ and that there are no incoming EMW's in the interaction region.

From (100), the amplitude of the outgoing wave can be calculated by inserting the expression for $F(z)$, expanding terms in $\Delta\omega \ll \omega$ and assuming that $k_l$ is approximately constant in the interaction region.

$$\begin{aligned} (\bar{E}_y)_{\mathrm{out}} &= -\frac{i}{2k_l}\left\{\omega^2\bar{h} - \frac{1}{2}i\omega\bar{h}\frac{i\Delta\omega^2}{2\omega}\right\}\int_0^L \hat{B}(z)\mathrm{e}^{\left(\frac{i\Delta\omega^2}{2\omega}\right)z}dz & (101) \\ &\approx -\frac{i\bar{h}\omega}{2}\frac{\sqrt{k_l}}{k_l}\int_0^L \hat{B}(z)\mathrm{e}^{\left(\frac{i\Delta\omega^2}{2\omega}\right)z}dz \\ &\approx -\frac{i\bar{h}\omega}{2}\int_0^L \hat{B}(z)\mathrm{e}^{\left(\frac{i\Delta\omega^2}{2\omega}\right)z}dz \quad \text{with} \quad \frac{\sqrt{k_l}}{k_l} \approx 1 & (102) \end{aligned}$$

where the second term in (101) follows from partial integration of $\frac{\partial\hat{B}}{\partial z}$ and the stockterm vanishes at the boundaries because the magnetic field is localized. The full solution for $\bar{E}_y(z,t)$ is then just:

$$\bar{E}_y(z,t) = (\bar{E}_y)_{\mathrm{out}}\mathrm{e}^{i(k_l z - \omega t)} \tag{103}$$

**Solution for $\hat{B}_x$:** The calculation of the magnetic part of the outgoing wave follows realizing that all that:

$$4\pi \frac{\partial}{\partial z}(\bar{j}_m)_y = 4\pi i k_l (\bar{j}_m)_y \approx 4\pi i\omega \left(1 - \frac{1}{2}\left(\frac{\Delta\omega}{\omega}\right)^2\right)(\bar{j}_m)_y \approx 4\pi i\omega(\bar{j}_m)_y \tag{104}$$

Comparing this with (95) and 96), it follows that $\hat{B}_x \approx -\hat{E}_y$, as one expects.

The EMWs, or 'light', generated by the gravitational waves are apparently hardly effected by the presence of a thin plasma. The wavenumber has a small offset with respect to the vacuum solutions, reflecting some dispersion caused by the plasma, but due to the strong magnetic background field, this will be a completely negligible effect. The wavenumber 'mismatch' is after all inversely proportional to the cyclotron frequency (in the regime $\omega_p^2/\omega_c \ll \omega \approx \omega_p \ll \omega_c$) and thus also to the magnetic field, so for a magnetic field of say $10^8$ Gauss and a typical frequency of a few kHz $(\Delta\omega)^2/\omega \sim 10^{-19}$rad/s.

## 6.5 Magnetars revisited

As mentioned in the previous section, the presence of a plasma in the interaction region is not very important. This is the result of the fact that the large magnetic field suppresses the mobility (resulting in $\Delta\omega \propto 1/\omega_c \propto 1/\hat{B} \ll 1$). Assuming therefore that the coherence length of the sc gw and EMW is larger than the length of the interaction region, the same results are found as for the vacuum case. The difference lies in the fact that in order to travel over astrophysical distances, the generated EMWs have to overcome the interstellar plasma. As soon as the magnetic field has decreased such that $\omega_c < \omega$, (89) and a spherical decay of the EMW beyond the interaction region allow the electron quiver velocity to become relativistic at a distance:

$$\begin{aligned} v(r) &\sim \frac{q}{m\omega}\frac{R_3}{r}\bar{E}_{\max} \\ r_{\rm rel} &\sim 3\cdot 10^{12}\text{km} \end{aligned} \tag{105}$$

and the electron velocity somewhere in between. In that case the EMWs become highly nonlinear and effects such as parametric excitation and, even more importantly, harmonic generation become important, which is interesting, because the latter effect can convert the EMWs to even larger frequencies.

It is not likely that this, non-linear, effect will play a very important rôle in the interstellar plasma, exciting higher harmonics with 60 times the frequency of the original waves, as has been done in laser experiments in a laboratory plasma. If, however, only the first or second higher harmonic is reached (with the same energy), this would be enough to overcome the interstellar plasma damping, and the resulting light should be detectable with a telescope such as the proposed *Astronomical Low Frequency Array* (ALFA). Supernovæ and collapsing binaries within the Local Group and the Virgo cluster might be detectable in this way with a event rate of as many as a few per year as soon as detectors such ALFA are operational. Furthermore, these signals could be compared with the GW observations when the first GW detectors such as LIGO, LISA and VIRGO open the GW observation chanel.

# 7 Magnetohydrodynamics

Another way to study interactions in a plasma is of course in the magnetohydrodynamic approximation. Consider an infinitely conductive hydrogen-like plasma in a singe fluid approximation (for more details, see [16] and [23]) where a pressure term is added in comparison to Sec. 6.3. In other words, consider the full energy-momentum tensor.

In an incompressible plasma, the unperturbed pressure will be negligible with respect to the energy density $\mu = \rho(1+\Pi)$ (with $\rho\Pi$ the, small, internal energy density), but the pressure gradients will be important. To avoid mistakes, all terms will be maintained throughout the exact calculation.

## 7.1 Conservation equations

The stress-energy tensor for a plasma in an electromagnetic field can naturally be decomposed in the tensor for the energy in the pure electromagnetic field that of the matter field, as in Sec. 6.3:

$$\begin{aligned} T^{ab} &= T^{ab}_m + T^{ab}_{\text{EM}} \qquad (106) \\ &= (\mu+p)V^aV^b + p\eta^{ab} + \frac{1}{4\pi}\left[F^{ac}F_c{}^b - \frac{1}{4}\eta^{ab}F^{cd}F_{cd}\right] \end{aligned}$$

which makes the conservation equations easy to evaluate (using the result of Exercise 3.18 in [18]):

$$\nabla_b T^{ab} = \nabla_b T^{ab}_m + \nabla_b T^{ab}_{\text{EM}} \qquad (107)$$

$$= \nabla_b T^{ab}_m - F^{ab}j_b = 0 \qquad (108)$$

Resulting, using the $3+1$ split, in the equation of energy conservation ($a=0$) and the equations of motion ($a=\alpha=1,2,3$). Again $V^a = (1,\mathbf{v})^T$ and $j^a = (\rho_m, \mathbf{j}_m)^T$.

$$\begin{aligned} \frac{\partial}{\partial t}(\mu+p) + \nabla\cdot(\mu+p)\mathbf{v} - \frac{\partial p}{\partial t} &= \mathbf{j}\cdot\mathbf{E} \qquad (109) \\ &- (\mu+p)\left[\Gamma^0{}_{db}V^dV^b + \Gamma^b{}_{db}V^d\right] \\ &- p\left[\Gamma^0{}_{db}\eta^{db} + \Gamma^b{}_{db}\eta^{0d}\right] \end{aligned}$$

$$\begin{aligned} \mathbf{v}\left(\frac{\partial}{\partial t}(\mu+p) + \nabla\cdot(\mu+p)\mathbf{v}\right) &+ (\mu+p)\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} + \nabla p \qquad (110) \\ &= \mathbf{j_m}\times\mathbf{B} - \rho\mathbf{E} - (\mathbf{v}\cdot\mathbf{j}_m)\mathbf{E} + (\mathbf{j}\cdot\mathbf{E})\mathbf{v} \\ &- (\mu+p)\left[\Gamma^\alpha{}_{db}V^dV^b + \Gamma^b{}_{db}V^\alpha V^d\right] \\ &- p\left[\Gamma^\alpha{}_{db}\eta^{db} + \Gamma^b{}_{db}\eta^{\alpha d}\right] \end{aligned}$$

Most terms in these rather long expressions vanish in a first order calculation in the metric of a plane polarised GW (79):

- All the rotation coefficients in the first equation vanish by themselves in this metric,
- the only non-vanishing Ricci coefficients in the second equation result in:

$$\frac{1}{2}i(\mu+p)(k_g\bar{v}_z+\omega_g)\bar{h}\begin{pmatrix} \bar{v}_x \\ -\bar{v}_y \\ 0 \end{pmatrix} \tag{111}$$

  which is clearly non-linear in the perturbed quantities,
- both $\mathbf{E}$ and $\mathbf{v}$ and $\mathbf{j}_m$ have no zeroth order components in the perturbation, so all the products of these quantities are second order or higher,
- $\rho\mathbf{E}$ is negligible with respect to $\mathbf{j_m}\times\mathbf{B}$ due to the charge neutrality in a plasma (the global charge density is very small, whereas the currents can be substantial; for details, again, see [16] or [23]),
- for the same reason the convective term $\mathbf{v}\cdot\nabla\mathbf{v}$ vanishes.

A final simplification is made by substituting the first equation in the second. What remains are the following simple conservation equations:

$$\begin{aligned} \frac{\partial\bar{\mu}}{\partial t}+(\mu+p)\nabla\cdot\bar{\mathbf{v}} &= 0 && (112)\\ (\mu+p)\frac{\partial\bar{\mathbf{v}}}{\partial t}+\nabla\bar{p} &= \bar{\mathbf{j}}_m\times\hat{\mathbf{B}} && (113)\\ &= \frac{1}{4\pi}(\nabla\times\bar{\mathbf{B}})\times\hat{\mathbf{B}}-\frac{1}{4\pi}\bar{\mathbf{j}}_E\times\hat{\mathbf{B}} && (114) \end{aligned}$$

where in the last line, 'Amperès' law is inserted without the displacement current, which is negligible with respect to the current $\mathbf{j}_m\sim\nabla\times\mathbf{B}\gg\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$ in the MHD approximation. The above equations are just the usual mass conservation equation and the equations of motion, but now with added terms that express the gravitational effects.

## 7.2 Ohms law

To close the system of equations, one also needs the generalized law of Ohm, which follows from the separate equation of motion for the electrons in the approximation that the electron inertia is negligible ($d\mathbf{v}_e/dt=0$). This is adequate as long as the perturbations caused by the GW are slow compared to the motion along the magnetic field (a weak constraint for the cyclotron motion around an extremely strong magnetic field), so the electrons have ample time to reach dynamical equilibrium. For zero resistivity, the single-fluid electron equation of motion is then given by (compare to (110) and (114)):

$$\nabla p_e=\rho_e(\mathbf{E}+\mathbf{v}_e\times\mathbf{B}) \tag{115}$$

or, from $\mathbf{j}_m \approx \rho_e(\mathbf{v}_i - \mathbf{v}_e)$ and therefore $\mathbf{v}_e \approx \mathbf{v} - \mathbf{v}_i \approx \mathbf{v} - \mathbf{j}/\rho_e$:

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\rho_e}(\mathbf{j} \times \mathbf{B} - \nabla p_e) \tag{116}$$

However, the more trivial and commonly known, Ohms law $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ prevails in the limit of a small Larmor radius with respect to the typical lenght scale of the GW perturbations ([23]). Since the magnetic field is very strong, this Larmor radius will be smaller than $10^{-8}$cm for electrons moving with thermal velocity ($v_{th,e} \approx 6 \cdot 10^7$m/s at $T \sim 10^8$K) and still smaller for lower perpendicular velocities. Even the smallest expectable GW perturbations should therefore be much larger than the Larmor radius.

### 7.3 MHD Maxwell equations

The reason for all this is to eliminate the electric field entirely from our set of equations, which is achieved by inserting the linearized, general law of Ohm in Faraday's equation:

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{v}} \times \hat{\mathbf{B}}) - \bar{\mathbf{j}}_B \tag{117}$$

**Linear gravity terms** The gravity induced current densities used in this equation and in (114) are to first order:

$$\begin{aligned} \mathbf{j}_E &= -\frac{1}{2}\hat{B}_x \frac{\partial}{\partial z} \begin{pmatrix} 0 \\ \bar{h} \\ 0 \end{pmatrix} \\ \mathbf{j}_B &= -\frac{1}{2}\hat{B}_x \frac{\partial}{\partial t} \begin{pmatrix} \bar{h} \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{118}$$

### 7.4 MHD wave dispersion relation

Finally, it is possible to derive a dispersion relation for the plasma waves, generated by the GWs that perturb the plasma. To obtain such an equation for the combined matter plus electrodynamic energy density content, the Faraday tensor no longer suffices, as it did in Sec. 4. What is needed now is the second covariant derivative of the total stress-energy tensor, which amounts to substituting (minus) the time derivative of the energy conservation equation (113) into the divergence of the equations of motion (114). This is somewhat similar to the procedure used when deriving magnetic wave equations by substituting the time derivative of Faraday's law into the curl of Ampères law (or the other way around to obtain electric waves) .

To first order, the wave equation becomes:

$$\begin{aligned}\nabla_a\nabla_b T^{ab} &= -\frac{\partial}{\partial t}\left(\nabla_b T^{0b}\right) + \nabla\cdot\left(\nabla_b T^{\alpha b}\right) \\ \frac{\partial^2\bar{\mu}}{\partial t^2} - \nabla^2\bar{p} &= -\frac{1}{4\pi}\nabla\cdot\left(\nabla\times\bar{\mathbf{B}}\right)\times\hat{\mathbf{B}} + \frac{1}{4\pi}\nabla\cdot\left(\bar{\jmath}_E\times\hat{\mathbf{B}}\right) \\ &= \frac{\hat{B}_x}{4\pi}\nabla^2\bar{\mathbf{B}} - \frac{\hat{B}_x^2}{8\pi}\nabla^2\bar{\mathbf{h}}\end{aligned} \tag{119}$$

or, with the definition of the sound velocity:

$$\left\{\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right\}\bar{\mu} = \frac{\hat{B}_x}{4\pi}\nabla^2\bar{\mathbf{B}} - \frac{\hat{B}_x^2}{8\pi}\nabla^2\bar{\mathbf{h}} \tag{120}$$

To eliminate $\bar{\mathbf{B}}$, differentiate (117) with respect to time and insert $\nabla\cdot\bar{\mathbf{v}}$ from the equation of mass conservation.

$$\begin{aligned}\frac{\partial^2\bar{\mathbf{B}}}{\partial t^2} &= -\hat{\mathbf{B}}\frac{\partial}{\partial t}\nabla\cdot\bar{\mathbf{v}} - \frac{\partial\bar{\jmath}_B}{\partial t} \\ &= \hat{\mathbf{B}}\frac{\partial^2}{\partial t^2}\left\{\frac{\bar{\mu}}{x} + \frac{\bar{h}}{2}\right\}\end{aligned} \tag{121}$$

Before continuing, first define the relativistic Alfvén speed by:

$$\frac{1}{u_A^2} = \frac{1}{c^2} + \left(\frac{\hat{B}_x^2}{4\pi x}\right)^{-1} \quad \text{or} \quad \frac{\hat{B}_x^2}{4\pi x} = \frac{u_A^2}{1 - u_A^2/c^2} \quad \text{with} \quad x = \mu + p \tag{122}$$

This expression reduces to $\frac{\hat{B}_x^2}{4\pi x} = u_A^2$ as long as we are discussing the MHD approximation, where the displacement current is negligible with respect to the current density. One has to keep in mind though, that in the limit of a very thin plasma ($\downarrow$ vacuum) with a strong magnetic field, the asymptotic behaviour of $u_A^2$ is that it goes to $c$ and of course not to infinite speed.

To arrive at the final dispersion relation for plasma waves excited by a incident gravitational wave, allow for damping, so the energy of the GW can dissipate into the

plasma. In other words, keep the frequency and wavenumber of the plasma waves unrestricted and try perturbations of the form:

$$\begin{aligned} h &= \bar{h}e^{i(k_g z - \omega_g t)} \\ \mu_{\text{tot}} &= \mu + \bar{\mu}e^{i(kz-\omega t)} \\ \mathbf{B}_{\text{tot}} &= \hat{\mathbf{B}} + \bar{\mathbf{B}}e^{i(kz-\omega t)} \end{aligned} \tag{123}$$

Inserting this into (120) and (121), results in the dispersion relations:

$$\{-\omega^2 + k^2(c_s^2 + u_A^2)\}\bar{\mu} = \frac{\hat{B}_x^2\omega_g^2}{8\pi}\left\{\frac{k_g^2}{\omega_g^2} - \frac{k^2}{\omega^2}\right\}\bar{h} \tag{124}$$

A more physical form of this equation is found after the left-hand-side of the equation is made dimensionless (just as the right-hand-side in $\bar{h}$) by normalizing the energy density perturbation to the total energy density: $\bar{\epsilon} = \bar{\mu}/x$.

Dividing both sides of (124) by $x$, leads to the final dispersion relation:

$$\boxed{\{-\omega^2 + k^2(c_s^2 + u_A^2)\}\bar{\epsilon} = \tfrac{1}{2}u_A^2\omega_g^2\left\{\tfrac{k_g^2}{\omega_g^2} - \tfrac{k^2}{\omega^2}\right\}\bar{h}}$$

The characteristics of this equation are discussed in the next subsection.

## 7.5 Summary

In this section, the equations of total energy density conservation and motion were derived in a tedrad system describing the influence of a passing gravitational wave on the underlying metric. It was found that these equations, to first order in $\bar{h}$, have exactly the same form as in a flat metric.

The coupling with the GW comes in through Maxwell's equations, used to eliminate **B**, with extra gravity induced terms. Standard MHD approximations of infinite conductivity and negligible displacement currents were used to eliminate **E** from the equations. Finally, a wave equation was derived from the second covariant derivative of the total energy-momentum tensor for a perfect fluid in a electromagnetic field.

The dispersion equation that resulted from this, satisfies the expected limiting behaviour:

- In the limit of vanishing GWs, $\bar{h} \to 0$, it reduces to a common fast magnetoacoustic plasma wave with $\omega^2 = k^2(c_s^2 + u_A^2)$.
- Without damping (when $k = k_g$ and $\omega = \omega_g$), the interaction disappears, and one finds the same magnetosonic solution.
- In the limit of vanishing density, $\bar{\epsilon} \to 0$, the displacement current is no longer neglegible, and one has to use the relativistic expressions for $u_A$ and $c_s$, that go to $c$ and 0 respectively in the vacuum limit. The 'plasma' dispersion relation therefore tends to $k = \omega$ (vacuum solution) and one retrieves the dispersion relation for a gravitational wave in vacuum, $k_g = \omega_g$.

- Finally, the strength of the coupling depends on the square of the magnetic background field through the Alfvén speed and on the square of the frequency of the driving GWs (compare to the linear dependence on the frequency of the amplitudes of the EMWs generated by GWs).

For a class of *gamma ray bursts*, powered by merging neutron star binaries, it would be very interesting if even a small fraction of the enormous amounts of energy released in gravitational waves could be dissipated into the surrounding plasma leading to the observed fireball af a GRB. This might be an alternative to the explanation that these fireballs are fueled purely by the neutrino flux from the merger, which has the complication that the continuous neutrino flux is already much larger than can be explained by the present models.

More work on this subject has to be done, though, before one can make any bold statements about the true importance of plasma waves generated by gravitational waves in the surroundings of merging neutron stars, leading to gamma ray burst fireballs. Obviously, the calculations should be extended to spherical symmetries, dipolar magnetic fields and spherically decaying gravitational waves. The EMWs calculated this way will probably look more like Bessel functions than plane waves. Still, the results of the first order calculations presented in this thesis seem significant enough to motivate further research.

# 8 Conclusions

In this thesis, several theoretical coupling effects between gravitational waves and electromagnetic waves were investigated. Some of these effects will be more significant than others in an astrophysical context.[7] The thesis started with an estimate of the general coupling between EMWs and GWs in an EM background field, through the Einstein field equations. The result of this, and of the subsequent exact calculation, was that the coupling efficiency depends on the square of the background field strength and the size of the interaction region, with however, and extremely small coupling constant ($\sim 10^{-50}(\mathrm{Ns}^2/\mathrm{kg\ m})^2$). These conversions therefore occur either in small regions with very strong EM background fields or in weaker fields extending over very large distances.

The astrophysical relevance of this process lies in three areas: it could provide indirect means to detect gravitational waves, it offers a possible explanation for the small fluctuations in the cosmic background radiation and it might prove to fuel the fireballs produced by gamma ray bursts. These results are discussed here in some detail.

## 8.1 Radio waves from binary mergers and magnetars

In the vicinity of merging neutron star binaries or quaking supernova remnant neutron stars surrounded by a vacuum, EMWs with maximum amplitudes of $10^{12}$V/m and 50MV/m respectively, could, in principle, be excited. The frequencies of these waves are $\sim 5$ kHz and $\sim 10$ kHz respectively, viz long radio waves, the latter of which might just be detectable with a space based radio array.

In the more realistic situation where the neutron stars are surrounded by a thin plasma, the light generated by the GWs obeys a slightly different dispersion relation, with $k_l^2 = \omega^2 - (\Delta\omega)^2$ instead of the vacuum plane wave dispersion relation $k = \omega$, where $\omega$ is the driving frequency of the GW. In other words, EMWs are still produced by the GWs, but there is some dispersion of these waves caused by the presence of a plasma. This dispersion shifts the wavenumber of the EMWs by a small amount and also effects the polarization by slightly changing the linear polarization relation, $\bar{B}_x = -\bar{E}_y$.

Both of these effects vanish, though, in the strong magnetic background fields under consideration. These fields suppress the electron mobility in the plasma and thereby also the dispersion caused by these electrons, resulting in $(\Delta\omega)^2 \uparrow 0$. What is left are, again, plane polarized radio waves with the same amplitudes as before in a vacuum.

The presence of a plasma does determine, however, whether the generated radio waves are able to travel over astronomical distances to the earth. The frequency of, in particular the EMWs coming from the binary mergers, is of the same order as the the interstellar plasma frequency. The radio waves will therefore be absorbed by the plasma, unless close to the merger, non-linear plasma effects result in socalled *photon acceleration*. This effect might lead to higher harmonics of the EMWs with the same energy, that might then be able to overcome the interstellar plasma. Light from magnetars could have high enough frequencies to overcome the plasma damping without such an additional effect.

[7] All of the mentioned couplings are *only* relevant in astrophysics.

From the most optimistic point of view, merging neutron star binaries and supernova remnants in our local galaxy and the Virgo cluster might, indirectly, be observable in radio waves generated by the large amounts of gravitational energy released in these processes. On the other hand, the highly idealized calculations (incident plane GWs etc.) have probably led to exaggerated results. More detailed (spherical) calculations are needed to examine what remains in more realistic situations.

## 8.2 Fluctuations in cosmic background radiation

Another interesting phenomenon that might be explained by the conversion of GWs to EMWs is the fluctuations in the 2.7 K microwave background radiation (GRB).

This radiation results from the decoupling era when the universe was approximately $20 \times 10^4$ years old. The temperature had then dropped to 2700 K resulting in the recombination of the until then, ionized matter. Because of this recombination, the diffusing 'electron mist' evaporated and became transparent to photons, leading to a present day, constant flux of thermal photons from this event.

The wavelength of these photons has reddened because of the expansion of the universe during their travel to earth and from this red-shift, one can determine that the size of the universe has increased with a factor 1000 since the decoupling. As the present size of the universe is supposed to be $10^{27}$ cm (from the redshift of quasars), one finds that it was $\sim 10^{24}$ cm when the CBR was created.

If, at that time, a primordial magnetic field of only $10^{-2}$ Gauss prevailed throughout space, a fraction of up to $10^{-3}$ of the GW energy could have been converted into EMW energy. According to popular cosmological models, this would be enough to explain the observed relative fluctuations of $10^{-5}$ in the cosmic background radiation, which would obviously be a very interesting result.

## 8.3 Gamma ray bursts

The most promising candidate as a source for socalled *gamma ray bursts* (GRBs) are merging neutron star binaries. These events can produce the required energy to fuel GRBs and moreover, they satisfy the temporal and size constraints set by the observations. Also, these mergers can be observed as often as once per day, which is comparable to the number of detected GRBs.

A problem in most GRB models is that the energy released by merging binaries is released primarily in GWs and not in EM radiation. The conversion of the energy in GWs to EM energy would therefore be very interesting. As was discussed extensively in this thesis, the direct conversion of GWs to light is not very effective, and more importantly, results in long radio waves, certainly not in gamma radiation.

A promising alternative is offered by the excitation of magnetohydrodynamic plasma waves, such as slow or fast magneto-acoustic waves and Alfvén waves. In Section 7.4, it was derived for the first time that, in particular, fast magnetosonic waves can indeed be generated through the interaction with gravitational waves passing through the plasma. The effectiveness of this process is proportional to the square of the strong magnetic

background field and the radio frequency of the perturbing GWs. In Appendix C it is shown, following [10], that non-linear GW effects can cause longitudinal Alfvén-like waves.

The magnetosonic waves are generated only when there is a wavelength (or, in the static case, a frequency) mismatch between the GW and the excited plasma waves. When this is the case, energy from the GW can dissipate into the plasma, which can later on emmit the energy in the form of EM (maybe gamma-) radiation.

Again, the work done in this thesis is not sufficient to fully describe what is going on in the vicinity of GRBs, but the result that a fraction of the GW energy from these sources could be converted into the observed EM radiation is one that deserves further research. Such research would, as a first improvement, have to include dipolar magnetic fields and spherical GW solutions for the merging neutron stars.

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Utrecht, march 2001.

# A Constants for exact GW to EMW calculation

**Region I**

$$
\begin{aligned}
A_I &= 0 \\
B_I &= (\mathrm{e}^{-2ikL}-1)\,[ \\
&+ \frac{1}{4}\left\{\frac{\alpha}{k^2} - B_z^{(0)}(\xi_{11}+\xi_{22}) - E_y^{(0)}(\xi_{00}-\xi_{22})\right\} \\
&+ \frac{1}{2}\left\{E_z^{(0)}\xi_{23} + B_x^{(0)}\xi_{30} - B_z^{(0)}\xi_{10} + E_x^{(0)}\xi_{12} + B_y^{(0)}\xi_{23})\right\}\,] \\
C_I &= 0 \\
D_I &= (\mathrm{e}^{-2ikL}-1)\,[ \\
&+ \frac{1}{4}\left\{\frac{\beta}{k^2} + B_y^{(0)}(\xi_{11}+\xi_{33}) - E_z^{(0)}(\xi_{00}-\xi_{33})\right\} \\
&+ \frac{1}{2}\left\{E_x^{(0)}\xi_{13} + B_y^{(0)}\xi_{10} - B_x^{(0)}\xi_{20}\right\}\,]
\end{aligned}
\tag{125}
$$

**Region II**

$$
\begin{aligned}
A_{II} &= \frac{1}{4}\left\{\frac{\alpha}{k^2} + B_z^{(0)}(\xi_{11}+\xi_{22}) - E_y^{(0)}(\xi_{00}-\xi_{22})\right\} \\
&+ \frac{1}{2}\left\{\frac{\alpha L}{ik} + E_z^{(0)}\xi_{23} + B_x^{(0)}\xi_{30} - B_z^{(0)}\xi_{10} + E_x^{(0)}\xi_{12} - B_y^{(0)}\xi_{23}\right\} \\
B_{II} &= -\frac{1}{4}\left\{\frac{\alpha}{k^2} - B_z^{(0)}(\xi_{11}+\xi_{22}) - E_y^{(0)}(\xi_{00}-\xi_{22})\right\} \\
&- \frac{1}{2}(E_z^{(0)}\xi_{23} + B_x^{(0)}\xi_{30} - B_z^{(0)}\xi_{10} + E_x^{(0)}\xi_{12} + B_y^{(0)}\xi_{23} \\
C_{II} &= \frac{1}{4}\left\{\frac{\beta}{k^2} - B_y^{(0)}(\xi_{11}+\xi_{33}) - E_z^{(0)}(\xi_{00}-\xi_{33})\right\} \\
&+ \frac{1}{2}\left\{\frac{\beta L}{ik} + E_x^{(0)}\xi_{13} + B_y^{(0)}\xi_{10} - B_x^{(0)}\xi_{20}\right\} \\
D_{II} &= -\frac{1}{4}\left\{\frac{\beta}{k^2} + B_y^{(0)}(\xi_{11}+\xi_{33}) - E_z^{(0)}(\xi_{00}-\xi_{33})\right\} \\
&- \frac{1}{2}\left\{E_x^{(0)}\xi_{13} + B_y^{(0)}\xi_{10} - B_x^{(0)}\xi_{20}\right\}
\end{aligned}
\tag{126}
$$

**Region III**

$$
\begin{aligned}
A_{III} &= \frac{\alpha L}{2ik} \\
B_{III} &= 0 \\
C_{III} &= \frac{\beta L}{2ik} \\
D_{III} &= 0
\end{aligned}
\tag{127}
$$

# B Symmetrizing the Maxwell equations

To prove that $(\epsilon^{\delta\alpha\gamma}\Gamma^{\beta}{}_{\delta\alpha} + \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\alpha})B_\gamma = \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\gamma}B_\alpha$ write:

$$\begin{aligned}
\epsilon^{\delta\alpha\gamma}\Gamma^{\beta}{}_{\delta\alpha}B_\gamma + \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\alpha}B_\gamma - \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\gamma}B_\alpha &= \\
(\eta^{\sigma\alpha}\epsilon^{\beta\delta\gamma} + \eta^{\sigma\beta}\epsilon^{\delta\alpha\gamma} - \eta^{\sigma\gamma}\epsilon^{\beta\alpha\delta})\Gamma_{\sigma\delta\alpha}B_\gamma &= \\
(\eta^{\sigma\alpha}\epsilon^{\gamma\beta\delta} + \eta^{\sigma\beta}\epsilon^{\alpha\gamma\delta} - \eta^{\sigma\gamma}\epsilon^{\beta\alpha\delta})\Gamma_{\sigma\delta\alpha}B_\gamma &= 0
\end{aligned} \tag{128}$$

So one has to prove that the expression in brackets is either symmetric in $\sigma\delta$ (since $\Gamma_{\sigma\delta\alpha}$ is skewsymmetric in these indices) or vanishes.

Distinguish between two cases:

**Case 1:** For $\alpha \neq \beta \neq \gamma$ the first term gives $\sigma = \delta$ and the other terms vanish,

**Case 2:** For $\alpha = \gamma \neq \beta$ the second term is zero and the other terms cancel.

**Conclusion** :

$$(\epsilon^{\delta\alpha\gamma}\Gamma^{\beta}{}_{\delta\alpha} + \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\alpha})B_\gamma = \epsilon^{\beta\delta\gamma}\Gamma^{\alpha}{}_{\delta\gamma}B_\alpha \tag{129}$$

which is nice, because it makes the inhomogeneous Maxwell equations symmetric with respect to the homogeneous ones.

# C Non-linear effects

It is to be expected that close to a source that emits strong GW's such as magnetars and neutron star binaries, non linear GW effects might become important. Consider, following [10], a plane fronted parallel, linearly polarized GW:

$$ds^2 = -dt^2 + a(z-t)^2 dx^2 + b(z-t)^2 dy^2 + dz^2 \tag{130}$$

with $z - t = u$ and $ab_{uu} + a_{uu}b = 0$. Using the same procedure as before, introduce an ONF, which in this case is a canonical Lorentz frame:

$$\begin{aligned} {e_{(0)}}^i &= (1,0,0,0)) \\ {e_{(1)}}^i &= (0,a^{-1},0,0) \\ {e_{(2)}}^i &= (0,0,b^{-1},0) \\ {e_{(3)}}^i &= (0,0,0,1) \end{aligned} \tag{131}$$

In this metric the gravity induced charge densities do not vanish, and the induced current densities have longitudinal components:

$$\begin{aligned} \rho_E &= -(\ln ab)_u E^3 \\ \rho_B &= -(\ln ab)_u B^3 \\ \mathbf{j}_E &= -(\ln b)_u (E^1 - B^2)\mathbf{e}_1 - (\ln a)_u (E^2 + B^1)\mathbf{e}_2 - (\ln ab)_u E^3 \mathbf{e}_3 \\ \mathbf{j}_B &= -(\ln b)_u (E^2 + B^1)\mathbf{e}_1 + (\ln a)_u (E^1 - B^2)\mathbf{e}_2 - (\ln ab)_u B^3 \mathbf{e}_3 \end{aligned} \tag{132}$$

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= -n(\ln ab)_u (1 - v_3) \\ \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} &= \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &+ ((\ln a)_u v_1 \mathbf{e}_1 + (\ln b)_u v_2 \mathbf{e}_2)(1 - v_3) + ((\ln a)_u v_1^2 + (\ln b)_u v_2^2)\mathbf{e}_3 \end{aligned} \tag{133}$$

If we assume that the GWs are approximately periodic and still have small amplitudes, the logarithmic factors that appear in these equations can be approximated by $a(u) = \sum_{n=-\infty}^{\infty} \hat{a}_n \exp in\omega u$ and $b(u) = \sum_{n=-\infty}^{\infty} \hat{b}_n \exp in\omega u$. To second order in $\bar{h}$ we find (see [10] for details):

$$\begin{aligned} (\ln a)_u &= i\omega\bar{h}e^{i\omega(z-t)} - \frac{1}{2}i\omega\bar{h}^2 e^{2i\omega(z-t)} + \text{c.c.} \\ (\ln b)_u &= -i\omega\bar{h}e^{i\omega(z-t)} - \frac{1}{2}i\omega\bar{h}^2 e^{2i\omega(z-t)} + \text{c.c.} \\ (\ln ab)_u &= -i\omega\bar{h}^2 e^{2i\omega(z-t)} - \frac{1}{8}i\omega\bar{h}^4 e^{4i\omega(z-t)} + \text{c.c.} \end{aligned} \tag{134}$$

For a GW moving to an initially unperturbed plasma with $\mathbf{E}_0 = \mathbf{B}_0 = \partial n_0/\partial t = 0$, (76) and (133), to first order, reduce to:

$$\begin{aligned}
\frac{\partial E_z}{\partial t} &= (\ln ab)_u E_z - 4\pi q n_0 v_z \\
\frac{\partial v_z}{\partial t} &= \frac{q}{m} E_z \qquad \text{and} \\
\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right) E &= \frac{\partial}{\partial t} (\ln ab)_u E
\end{aligned} \tag{135}$$

Obviously, the flat space plasma oscillations are changed by the GW, especially at the resonance frequency $\omega_p = \omega$, where the parametric excitation of longitudinal plasma oscillations is given by:

$$\frac{d\bar{E}}{dt} = -\frac{1}{2} i\omega \bar{h}^2 \mathrm{e}^{2i\omega z} \bar{E}^* \tag{136}$$

with a growth rate of:

$$\Gamma = \frac{1}{2}\omega |\bar{h}^2| \tag{137}$$

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